M.Tech I[st] YEAR I SEMISTER UNIT- I

THEORY OF ELASTICITY

Elasticity - Notation for forces & Stresses - Components of stresses and Strains - Hooke's Law - Plane stress - Plane strain - Differential Equations of Equilibrium - Boundary Conditions - Compatibility Equations - Stress function - Boundary conditions.

* Elasticity:-

If the External forces producing deformation do not Exceed a certain limit, the deformation disappears with the removal of the forces.

① Atomic Structure will not be Considered here

② It will be Assumed that the matter of elastic body is "homogeneous" & Continously distributed over its volume so that the smallest element cut from the body possesses the same Specific physical properties as the body.

③ To ~~Simplify the~~ Simplify the discussion it will also be assumed that for the most part the body is "Isotropic", i.e., that the elastic properties are the same in all directions.

* Notations for forces & Stresses:-

① In general case the direction of stress ~~is inclined~~ can resolve into two components:

→ normal Stress perpendicular to the area

→ shearing stress acting in the plane of the area ~~δA~~

② There are two kinds of External forces which may act on bodies

→ Surface forces

→ body forces

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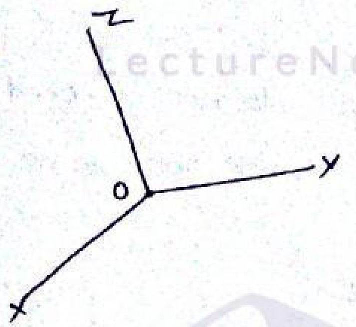
③ The surface force per unit area we shall resolve into three components & they are notated by $\hat{X}, \bar{Y}, \bar{Z}$.

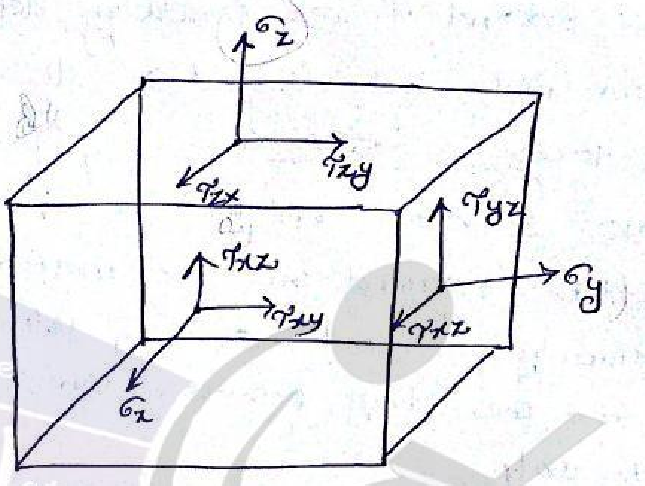
④ We shall also resolve the body force per unit volume into three components & they are notated by X, Y, Z.

⑤ Normal stress can be designated by "σ"

Shear stress " " "τ"

⑥

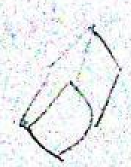




⑥ For the sides of the element perpendicular to the y axis, for instance, the normal components of stress acting on these sides denoted by σ_y, subscript y indicates that the stress is acting on a plane normal to the Y-axis.

⑦ The shearing stress is resolved into two components parallel to the co-ordinate axes.

⑧ Two subscript letters are used in this case, the first indicating the direction of the normal to the plane under consideration & the second indicating the direction of the component of the stress.



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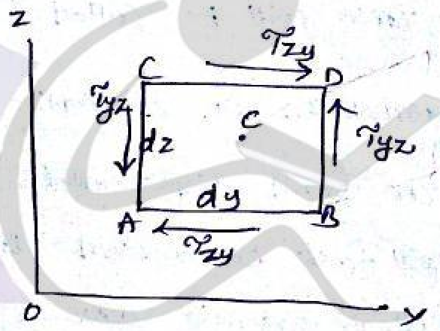
Components of Stress:-

① From the discussion, we see that for each Pair of Parallel sides of a cubic element, Such as One symbol is needed to denote the Normal Component of stress & two more Symbols to denote the two Components of Shearing stress.

② To describe the Stresses acting on the Six sides of the element three Symbols $\sigma_x, \sigma_y, \sigma_z$ are necessary for Normal Stresses & Six Symbols $\tau_{xy}, \tau_{yx}, \tau_{zx}, \tau_{yz}, \tau_{zy}$ for Shearing Stresses.

③ By a Simple Consideration of the Equilibrium of the element the no. of Symbols for shearing Stresses Can be reduced to three.

④ Body forces, Such as the weight of the element, ~~the body forces acting~~ can be neglected in this instance because in reducing the dimensions of the element the body forces acting on it diminish as the cube of the linear dimensions, Where as the Surfaces forces diminish as the square of the linear dimensions.



⑤ Hence, for a very small element, body forces are small quantities of higher order than Surface forces & can be omitted in calculating moments.

⑥ Similarly, moments due to non-Uniformity of distribution of normal forces are of higher order than those due to the Shearing forces & vanish in the limit.

⑦ Also forces on each side can be considered to be the area of the side & times the stress at the middle.

⑧ i.e, the force on AB side is $\tau_{zy} \times dy \times dx$

on BD side is $\tau_{yz} \times dz \times dx$.

⑨ taking moments of forces about "C"

$$(\tau_{zy} \times \cancel{dy} \times \cancel{dx})\, \cancel{dz} = (\tau_{yz} \times \cancel{dz} \times \cancel{dx})\, \cancel{dy}$$

$$\boxed{\tau_{zy} = \tau_{yz}}$$

i.e, $\tau_{zy} = \tau_{yz}$; $\tau_{xz} = \tau_{zx}$; $\tau_{xy} = \tau_{yx}$

⑩ The six quantities $\sigma_x, \sigma_y, \sigma_z, \tau_{zy} = \tau_{yz}, \tau_{xy} = \tau_{yx}, \tau_{xz} = \tau_{zx}$ are therefore sufficient to describe the stresses acting on the co-ordinate planes through a point.

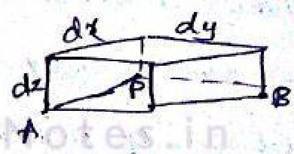
these will be called "Components of stress" at the point.

* COMPONENTS OF STRAIN:-

① the small displacements of particles of a deformed body will first be resolved into components u, v, w parallel to the co-ordinate axes x, y, z respectively.

② Consider a small element dx, dy, dz of an elastic body

③ if the body undergoes a deformation & u, v, w are the components of the displacement of the point 'P', the displacement in the x direction of an adjacent point A on the x axis is, to the first order in dx

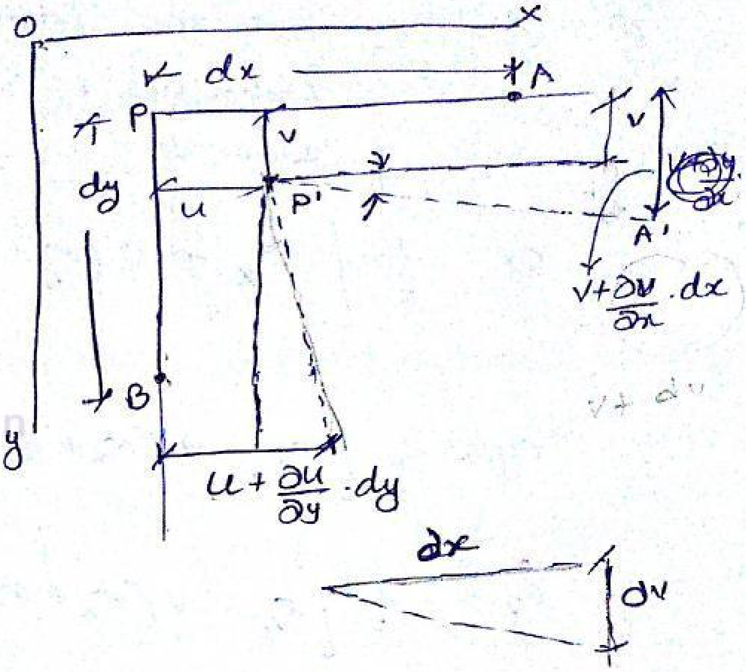


$$u + \frac{\partial u}{\partial x} \cdot dx$$

④ due to the increase $\left(\frac{\partial u}{\partial x}\right) dx$ of the function i.e with increase of the co-ordinate x.

⑤ The increase in length of the element PA due to deformation is therefore $(\partial u / \partial x)\, dx$.

6. Hence the unit elongation at point P in the x direction is $(\partial u/\partial x)$.



7. if u & v are the displacements of the point "P" in the x & y directions, the displacement of the point A in the y direction & the B in the x direction are $v + \left(\frac{\partial v}{\partial x}\right) dx$ & $u + \frac{\partial u}{\partial y} dy$ respectively

8. Owing to these displacement the new direction P'A' of the element PA is inclined to the initial direction by the small angle indicated $\partial v / \partial x$

9. from this it will be seen that the initially right angle APB between the two elements PA & PB is diminished by the angle $\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$. This is the shearing strain b/w xz & yz

10. Consider γ is the letter for unit shearing strain ϵ for unit elongation.

$$\left.\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} \quad \epsilon_y = \frac{\partial v}{\partial y} \quad \epsilon_z = \frac{\partial w}{\partial z} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \end{aligned}\right\} \rightarrow (2)$$

The six components of are called "Components of Strain".

* HOOKE'S LAW:-

① $\varepsilon_x = \frac{\sigma_x}{E}$ but the extension of the element in the x-direction is accompanied by lateral strain components

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right]$$

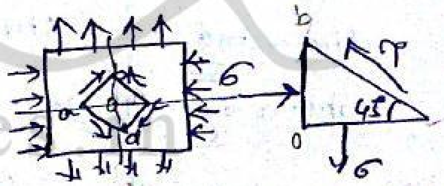
$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] \longrightarrow (3)$$

ν: poisson's ratio usually for structural steel $\nu = 0.30$.

② Eqn 3, the relations b/w elongations & stresses are completely defined by two physical constants E & ν.

③ The same constants can also be used to define the relation b/w shearing strain & shearing stress.

④ Let us consider the particular case of deformation of rectangular parallelepiped in which



$\sigma_z = \sigma$, $\sigma_y = -\sigma$, $\sigma_x = 0$ (Cutting out an element abcd by planes parallel to the x axis & at 45° to the y & z axes.

⑤ ~~the~~ by summing up the forces along & perpendicular to bc, that the normal ~~force~~ stress on the sides of this element is zero.

$\cancel{\tau^2 = \sqrt{\sigma^2+\sigma^2} = \sqrt{2}\sigma}$

$$(\tau\, bc)^2 = (ob\,\sigma)^2 + (\sigma\, oc)^2$$

$$bc^2\tau^2 = ob^2\sigma^2 + oc^2\sigma^2$$

$$bc\,\tau = \sigma\sqrt{ob^2 + oc^2}$$

$\cancel{\tau = \frac{\sigma_z}{2} + \left(\frac{\sigma_y}{2}\right)}$

$\cancel{\tau = \frac{\sigma}{2} + \frac{\sigma}{2} = \sigma}$

$$\boxed{\tau = \sigma}$$

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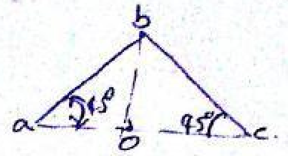
(6) The angle b/w the sides ab & bc changes & the corresponding magnitude of shearing strain γ may be found from the obc triangle after deformation



$$\frac{Oc}{Ob} = \tan\left(\frac{\pi}{4} - \frac{\gamma}{2}\right)$$

(7) The relation b/w shearing strain & shearing stress is defined as

$$G = \frac{E}{2(1+\nu)} \qquad \boxed{\gamma = \frac{\tau}{G}}$$

G = Modulus of elasticity in shear (or) modulus of rigidity

$$\boxed{\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz} \qquad \gamma_{xz} = \frac{1}{G}\tau_{zx}} \rightarrow 3(a)$$

Suppose the components of stress expressed as functions of the components of strain are needed then

$$e = \epsilon_x + \epsilon_y + \epsilon_z \rightarrow (1) \quad ; \quad \theta = \sigma_x + \sigma_y + \sigma_z \rightarrow (2)$$

the relation b/w volume expansion e & the sum of normal stresses

$$e = \frac{1-2\nu}{E}\theta \Rightarrow \boxed{\theta = \frac{Ee}{1-2\nu}}$$

from eq (2)

$$\theta - \sigma_x = \sigma_y + \sigma_z$$

$$\epsilon_x = \frac{1}{E}\left(\sigma_x - \nu(\sigma_y + \sigma_z)\right)$$

$$\frac{Ee}{1-2\nu} - \sigma_x = \sigma_y + \sigma_z$$

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z)$$

$$\frac{Ee}{1-2\nu} - \sigma_x = \frac{\sigma_x}{\nu} - \frac{\epsilon_x E}{\nu}$$

$$\boxed{\frac{\epsilon_x E}{\nu} - \frac{\sigma_x}{\nu} = \sigma_y + \sigma_z}$$

$$\frac{Ee}{1-2\nu} + \frac{\epsilon_x E}{\nu} = \frac{\sigma_x}{\nu} + \sigma_x$$

$$\frac{Ee}{1-2\nu} + \frac{\epsilon_x E}{\nu} = \left(\frac{1+\nu}{\nu}\right)\sigma_x$$

$$\sigma_x = \frac{\nu}{1+\nu}\left[\frac{Ee}{1-2\nu}\right] + \frac{\nu}{1+\nu} \times \frac{\varepsilon_x E}{\nu}$$

$$\boxed{\sigma_x = \frac{Ee}{(1+\nu)(1-2\nu)}\nu + \frac{E\varepsilon_x}{1+\nu}} \rightarrow (4)$$

$$\boxed{\sigma_y = \frac{Ee}{(1+\nu)(1-2\nu)}\nu + \frac{E}{1+\nu}\varepsilon_y} \rightarrow (5)$$

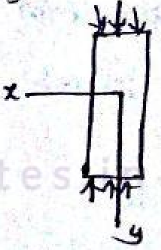
$$\boxed{\sigma_z = \frac{Ee}{(1+\nu)(1-2\nu)}\nu + \frac{E}{1+\nu}\varepsilon_z} \rightarrow (6)$$

if $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$ $\qquad 2G = \frac{E}{1+\nu}$

$$\boxed{\begin{aligned} \sigma_x &= \lambda e + 2G\varepsilon_x \\ \sigma_y &= \lambda e + 2G\varepsilon_y \\ \sigma_z &= \lambda e + 2G\varepsilon_z \end{aligned}} \rightarrow (7)$$

* PLANE STRESS:-

if a thin plate is loaded by forces applied at the boundary, parallel to the plane of the plate & distributed uniformly over the thickness, the stress components $\sigma_z, \tau_{xz}, \tau_{yz}$ are zero on both faces of the plate, & it may be assumed, tentatively, that they are also with to the plate.



The state of stress is then specified by $\sigma_x, \sigma_y, \tau_{xy}$ only & is called plate stress.

⇒ it may also be assumed that 3 components are independent of z i.e., they do not vary through the thickness.

⇒ They are then functions of x & y only.

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* PLAIN STRAIN:-

① A Similar Simplification is possible at the other Extreme when the dimension of the body in the z direction is very large.

② if a long cylindrical (or) prismatical body is loaded by forces that are perpendicular to the longitudnal elements & do not vary along the length, it may be assumed that all cross sections are in the same condition.

③ it is simplest to suppose at first that the end sections are confined b/w fixed smooth rigid planes, so that the displacement in the axial direction is prevented.

④ There are many important problems of this kind, for instance, a retaining wall with lateral pressure, a culvert. (or) tunnel, a cylindrical tube with internal pressure

⑤ The components u & v of the displacement are functions of x & y but are independent of the longitudnal co-ordinate z.

⑥ Since longitudnal displacement w is zero

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \;; \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0$$

$$\varepsilon_z = \frac{\partial w}{\partial z} = 0$$

⑦ The longitudnal normal stress σ_z can be found in terms of σ_x & σ_y: Since $\varepsilon_z = 0$

$$\sigma_z - \nu(\sigma_x + \sigma_y) = 0$$

$$\boxed{\sigma_z = \nu(\sigma_x + \sigma_y)}$$

⑧ These normal stresses act over the cross sections including the ends, where they represent forces required to maintain the plane strain & provided by the fixed smooth rigid planes.

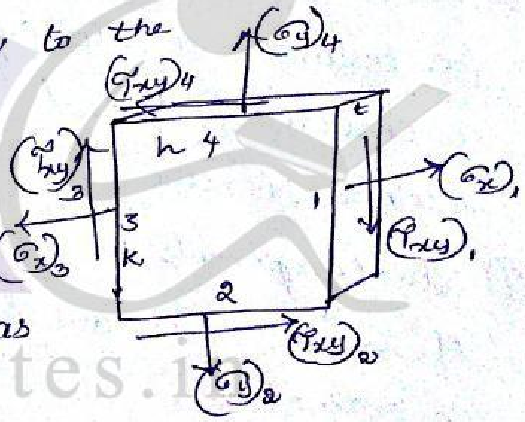
⑨ the stress components τ_{xz} & τ_{yz} are zero. Thus the plane strain problem, like the plane stress problem, reduces to the determination of σ_x, σ_y & τ_{xy} as functions of x & y only.

* Differential Equations of Equilibrium:-

① Consider the equilibrium of a small block of edges h, k, t.

② These stresses acting on the faces 1, 2, 3, 4 &

③ The symbols $\sigma_x, \sigma_y, \tau_{xy}$ refer to the point x, y the mid point of block.



④ The body forces on the block, which was neglected as a small quantity.

⑤ if X, Y denote the components of body force per unit volume, the Equation of equilibrium for forces in x direction.

$$(\sigma_x)_1 kt - (\sigma_x)_3 kt + (\tau_{xy})_2 h \times t - (\tau_{xy})_4 ht + Xhkt = 0$$

by dividing $h k t$

$$\frac{(\sigma_x)_1 - (\sigma_x)_3}{h} + \frac{(\tau_{xy})_2 - (\tau_{xy})_4}{k} + X = 0$$

⑥ if now the block is taken smaller & smaller, that is $h \to 0$, $k \to 0$

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the limit of $\frac{(\sigma_x)_1 - (\sigma_x)_3}{h}$ is $\partial \sigma_x / \partial x$

Ho & $\frac{(\tau_{xy})_2 - (\tau_{xy})_4}{K}$ is $\partial \tau_{xy}/\partial y$

Thus

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + Y = 0 \end{aligned}\right\} \quad (3)$$

These are the differential Equations of Equilibrium for two dimensional problems.

In Many practical applications the wt. of the body is usually the only body force. Taking the y-axis downward & denoting by ρ the mass per unit volume of the body Eqn 3 becomes

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad (4) \quad ; \quad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \rho g = 0 \quad (5)$$

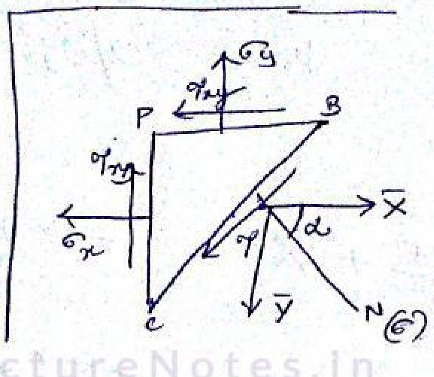
* Boundary Conditions:-

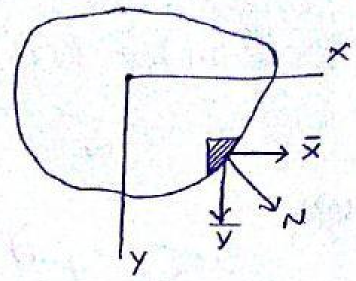
1) These stress components Vary over the Volume of the plate

2) when we arrive at the boundary they must be such as to be in equilibrium with the External forces on the boundary of plate, So that the External forces may regarded as a Continuation of the Internal Stress distribution.

3) Taking the Small traingular Prism PBC, so that the Side BC coincides with the boundary

of the plate & denoting by $\bar{X}$ & $\bar{Y}$





in which l & m are the direction cosines of the Normal N to the boundary.

$$\boxed{\begin{aligned}\bar{X} &= l\sigma_x + m\tau_{xy}\\ \bar{Y} &= m\sigma_y + l\tau_{xy}\end{aligned}}$$

(4) Taking, for instance, a side of the plate parallel to the x-axis; hence $l=0$; $m=\pm 1$

the $\bar{X} = \pm \tau_{xy}$; $\bar{Y} = \pm \sigma_y$

(5) It is seen that at the boundary the Stress Components become equal to the components of the surface forces per unit area of the boundary.

* COMPATIBILITY EQUATIONS:-

(1) It is the fundamental problem of the theory of elasticity to det. the State of Stress in a body Submitted to the action of given forces

(2) In 2-dimensional problem it is necessary to solve the differential equations of Equilibrium & the solution must be such as to Satisfy the boundary Conditions.

(3) These equations, derived by application of the equations of statics & Containing 3 stress components σ_x, σ_y, τ_{xy} are not Sufficient for the determination of these Components.

(4) The problem is a statically indeterminate one, & in order to obtain the solution the elastic deformation of the body must also be considered.

$$\varepsilon_x = \partial u/\partial x \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

the relation b/w strain components can be obtained from

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \cdot \partial y} \longrightarrow (10)$$

This D.E is called "Condition of Compatibility"

(5) By using Hooke's law equation can be transformed into a relation b/w the components of stress.

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} = \frac{1}{\frac{E}{(1+\nu)2}} \times \tau_{xy}$$

In plane stress

$$G = \frac{E}{(1+\nu)2}$$

$$\frac{\partial^2}{\partial y^2}(\sigma_x - \nu\sigma_y) + \frac{\partial^2}{\partial x^2}(\sigma_y - \nu\sigma_x) = 2(1+\nu)\cdot\frac{\partial^2 \tau_{xy}}{\partial x \cdot \partial y} \rightarrow (6)$$

From eq (4) & (5)

$$\frac{\partial \sigma_x}{\partial x} = \frac{-\partial \tau_{xy}}{\partial y} \;; \quad \frac{\partial \sigma_y}{\partial y} = \frac{-\partial \tau_{xy}}{\partial x}$$

$$\frac{\partial^2 \sigma_x}{\partial x^2} = \frac{-\partial^2 \tau_{x}}{\partial x \cdot \partial y} \;; \quad \frac{\partial^2 \sigma_y}{\partial y^2} = \frac{-\partial^2 \tau_{xy}}{\partial x \cdot \partial y}$$

$$\& \quad \frac{\partial^2 \tau_{xy}}{\partial x \cdot \partial y} = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2}$$

Eq ⑥ Substitute $2\frac{\partial^2 \tau_{xy}}{\partial x \cdot \partial y}$ value.

$$\frac{\partial^2}{\partial y^2}\sigma_x - \frac{\partial^2}{\partial y^2}\nu\sigma_y + \frac{\partial^2}{\partial x^2}\sigma_y - \frac{\partial^2}{\partial x^2}\nu\sigma_x = 2\frac{\partial^2 \tau_{xy}}{\partial x \cdot \partial y} +$$

$$2 \times \frac{\partial^2 \tau_{xy}}{\partial x \cdot \partial y}$$

$$\frac{\partial^2}{\partial y^2}\sigma_x - \frac{\partial^2}{\partial y^2}\nu\sigma_y + \frac{\partial^2}{\partial x^2}\sigma_y - \frac{\partial^2}{\partial x^2}\nu\sigma_x = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2}$$

$$-\nu\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y}\right)$$

$$\frac{\partial^2}{\partial y^2}\sigma_x + \frac{\partial^2}{\partial x^2}\sigma_y + \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} = 0$$

$$\boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0} \longrightarrow ⑦$$

proceeding to the same manner from eqn ③

$$\boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)}$$ this is also

2nd method

* In plane strain problem $\sigma_z = \nu(\sigma_x + \sigma_y)$

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu(\sigma_y + \sigma_z)\right)$$

$$= \frac{1}{E}\left(\sigma_x - \nu(\sigma_y + \nu\sigma_x + \nu\sigma_y)\right)$$

$$= \frac{1}{E}\left(\sigma_x - \nu\sigma_y - \nu^2\sigma_x - \nu^2\sigma_y\right)$$

$$\boxed{\varepsilon_x = \frac{1}{E}\left[\sigma_x(1-\nu^2) - \nu(1+\nu)\sigma_y\right]}$$

$$\varepsilon_y = \frac{1}{E}\left[(1-\nu^2)\sigma_y - \nu(1+\nu)\sigma_x\right]$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$$

from Eqⁿ ⑧ & ⑯ substitute the "ε" values we can det; the body forces

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_x+\sigma_y) = -\frac{1}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)$$

* <u>Stress function:-</u>

actually
$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0$$

$$\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}+\rho g=0$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\sigma_x+\sigma_y=0$$

To these equations the B.C $\bar{X} = \pm \tau_{xy}$, $\bar{Y} = \pm \sigma_y$ should be added.

⇒ The general method of solving these eqns by introducing a new function called "Stress function".

⇒ As easily checked taking any function ϕ of x & y and putting the following Expressions for stress components.

$$\left.\begin{aligned}\sigma_x &= \frac{\partial^2\phi}{\partial y^2}-\rho g y \\ \sigma_y &= \frac{\partial^2\phi}{\partial x^2}-\rho g y \\ \tau_{xy} &= \frac{-\partial^2\phi}{\partial x\,\partial y}\end{aligned}\right\} \rightarrow (29)$$

In this manner we can get a variety of Solutions of the Equ of equilibrium.

The true Solution which Satisfy the Compatibility Equation

$$\frac{\partial^4\phi}{\partial x^4}+2\cdot\frac{\partial^4\phi}{\partial x^2\,\partial y^2}+\frac{\partial^4\phi}{\partial y^4}=0$$

UNIT - II

Two dimensional problems in Rectangular Co-ordinates - solution by polynomials - Saint Venant's Principle - Determination of Displacements - Bending of Simple beams - Application of fourier Series for two dimensional problems for gravity loading.

* Solution by polynomials:-

the solution of 2-dimensional problems, when the body forces ~~ab~~ are absent then the D.E

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \cdot \frac{\partial^4 \phi}{\partial x^2 \cdot \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \longrightarrow (1)$$

polynomial of Second degree:-

$$\phi_2 = \frac{a_2}{2} x^2 + b_2 xy + \frac{c_2}{2} y^2$$

actually the stress Components from Stress function

$$\boxed{\rho g y = 0}$$

$$\sigma_x = \frac{\partial^2 \phi_2}{\partial y^2} - \rho g y$$

$$= \frac{\partial^2}{\partial y^2}\left(\frac{a_2 x^2}{2} + b_2 xy + \frac{c_2}{2} y^2\right)$$

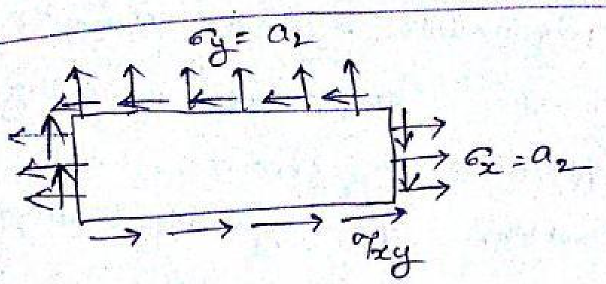
$$\boxed{\sigma_x = c_2} \qquad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} - \rho g y = a_2$$

$$\boxed{\sigma_y = a_2}$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \cdot \partial y} = -b_2 \qquad \boxed{\tau_{xy} = -b_2}$$

All three ~~Sa~~ stress Components are constant throughout the body.

i.e., the stress function represent a combination of Uniform tensions (or) compressions in 2 perpendicular & Uniform shear.



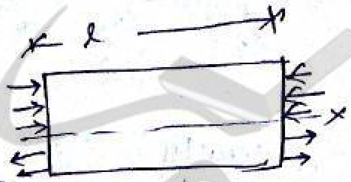
* Polynomial of third degree (Pure bending (or) tensile & shearing stress)

$$\phi_3 = \frac{a_3}{2}\times\frac{x^3}{3} + \frac{b_3 y}{2}x^2 + \frac{c_3}{2}xy^2 + \frac{d_3}{2}\times\frac{y^3}{3}$$

$$\sigma_x = \frac{\partial^2\phi_3}{\partial y^2} = c_3 x + d_3 y$$

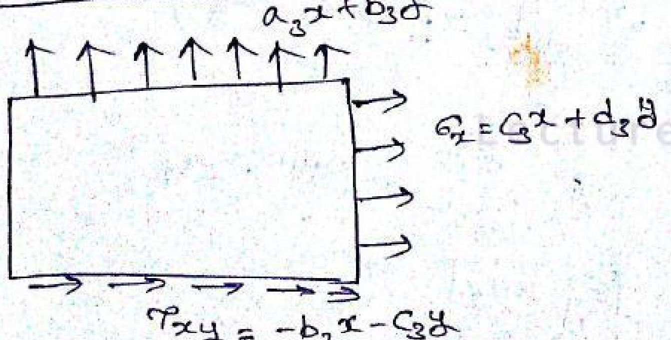
$$\sigma_y = a_3 x + b_3 y$$

$$\tau_{xy} = -\frac{\partial^2\phi_3}{\partial x\,\partial y} = -b_3 x - c_3 y$$

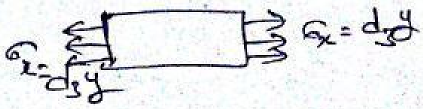


⇒ for a rectangular plate, assuming all co-efficients except d_3 equal to zero, we obtain pure bending.

⇒



i.e., we have only normal stress acting on side "y"



⇒ if the co-efficient b_3 (or) c_3 is taken different from zero then we obtain not only normal but also shear stresses acting on sides of plate.

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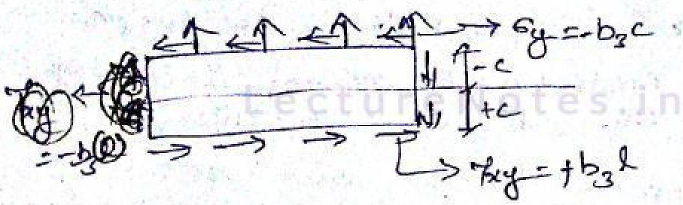
⇒ In case all co-efficients Except b_3 are equal to zero

$\boxed{\text{We have Uniformly distributed tensile \& stress \& shearing stress } \propto x}$

Suppose if side $x = l$ we have shearing stress $-b_3 l$

Suppose if side $x = 0$; there are no stress acting on x side



* Taking the stress function In the form of a polynomial of fourth degree (Pure shear, σ_x on one side)

$$\phi_4 = \frac{a_4 x^4}{4(3)} + \frac{b_4 x^3 y}{3(2)} + \frac{c_4 x^2 y^2}{2} + \frac{d_4 x y^3}{3(2)} + \frac{e_4 y^4}{(4)(3)}$$

this eqn is satisfied only if $\boxed{e_4 = -(2c_4 + a_4)}$ ***

Stress Components

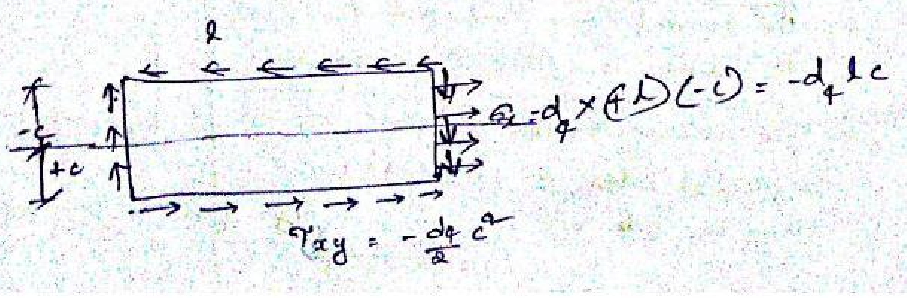
$$\sigma_x = \frac{\partial^2 \phi_4}{\partial y^2} = c_4 x^2 + d_4 xy - (2c_4 + a_4) y^2$$

$$\sigma_y = a_4 x^2 + b_4 xy + c_4 y^2$$

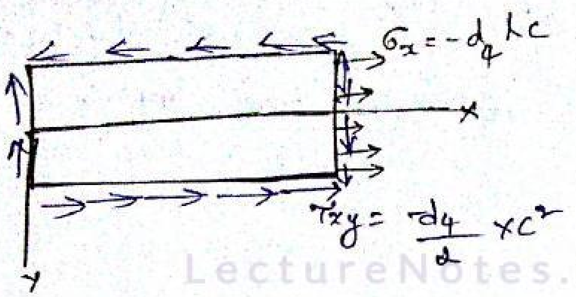
$$\tau_{xy} = -\frac{b_4}{2} x^2 - 2c_4 xy - \frac{d_4}{2} y^2$$

for taking all co-efficients Except d_4 equal to zero then

$$\sigma_x = d_4 xy; \quad \sigma_y = 0; \quad \tau_{xy} = \frac{-d_4}{2} y^2$$



* the shearing forces acting on the boundary of the plate reduce to the couple



* the shearing forces acting on the boundary of the plate reduce the couple.
* This couple balances the couple produced by the normal forces along the side x=l of the plate.

* Let us consider a stress function in the form of polynomial of the fifth degree (σ_y on two sides, σ_x on one side, σ_{xy} on 3 sides)

$$\phi_5 = \frac{a_5}{5(4)} x^5 + \frac{b_5}{4(3)} x^4 y + \frac{c_5}{3(2)} x^3 y^2 + \frac{d_5 x^2 y^3}{3(2)} + \frac{e_5 x y^4}{4(3)} + \frac{f_5 y^5}{5(4)}$$

$$e_5 = -(2c_5 + 3a_5)$$

$$f_5 = -\frac{1}{3}(b_5 + 2d_5)$$

$$\boxed{\sigma_x = \frac{\partial^2 \phi_5}{\partial y^2} = d_5\left(x^2 y - \frac{2}{3} y^3\right)}$$

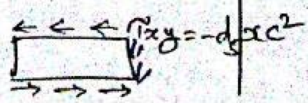
$$\boxed{\sigma_y = \frac{1}{3} d_5 y^3} \qquad \boxed{\tau_{xy} = -d_5 x y^2}$$

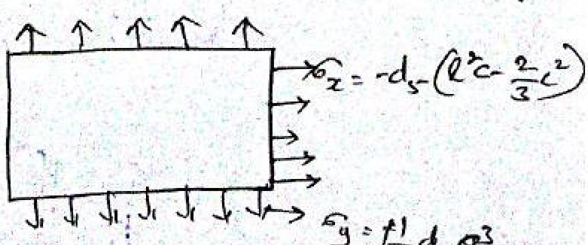
Taking for instance, all co-efficients, except d_5 equal to zero

⇒ The normal forces are uniformly distributed along the longitudinal sides of the plate

⇒ the normal force consists of two parts, 1 is linear law, 2 is cubic parabola

⇒ The shearing forces are $\propto x$ on the longitudinal sides of parabolic law along the side x=l





superposition will considered.

(5) In taking the stress function in the form of polynomials (2)
of the second & third degrees we are completely free in choosing the magnitudes of the co-efficients.

(6) In case of polynomials of higher degrees certain relations b/w the co-efficients are satisfied.

the stress function in the form of polynomial of 4th degree

$$\phi_4 = \frac{a_4}{4(3)}x^4 + \frac{b_4}{3(2)}x^2y + \frac{c_4}{2}x^2y^2 + \frac{d_4}{3(2)}xy^3 + \frac{e_4}{4(3)}y^4$$

we find the eqn is satisfied only if.

$$e_4 = -(2c_4 + a_4)$$

$$\sigma_x = \frac{\partial^2\phi_4}{\partial y^2} = c_4x^2 + d_4xy - (2c_4 + a_4)y^2$$

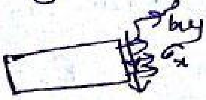
$$\sigma_y = \frac{\partial^2\phi_4}{\partial x^2} = a_4x^2 + b_4xy + c_4y^2$$

$$\tau_{xy} = \frac{-\partial^2\phi_4}{\partial x\,\partial y} = -\frac{b_4}{2}x^2 - 2c_4xy - \frac{d_4}{2}y^2$$

For instance taking all co-efficients except d_4 equal to zero we obtain pure shear

$\boxed{\sigma_x = d_4xy}$; $\boxed{\sigma_y = 0}$; $\boxed{\tau_{xy} = -\frac{d_4}{2}y^2}$

diagram



(7) the shearing forces acting on the boundary of the plate reduce to the couple

then the stress components are

$$\sigma_x = d_5\left(x^2y - \frac{2}{3}y^3\right)$$

$$\sigma_y = \frac{1}{3}d_5y^3$$

$$\tau_{xy} = -d_5xy^2$$

Taking except d_5 equal to zero we find the normal forces are uniformly distributed along the longitudinal sides.

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* SAINT - VENANT'S PRINCIPLE :-

① We have several solutions for rectangular plates were obtained from very simple forms of the stress function ϕ.

② Many solutions have been obtained not only for rectangular regions but for prismatic, cylindrical & tapered shapes.

③ These show that a change in the distribution of the load on an end, without change of the resultant. In such cases, simple solutions give sufficiently accurate results except near the ends.

④ "The change of distribution of the load is equivalent to the superposition of a system of forces statically equivalent to zero force & zero couple" is called "Saint Venant's principle".

* Determination of Displacements :-

① Actually the displacement can be obtained from hooke's law

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] \cdots\cdots$$

but we can also obtained displacements by

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad ; \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad ; \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

② It may be seen ~~that~~ once that the strain components remain unchanged if we add to u & v

$$u_1 = a + by \qquad v_1 = c - bx \qquad (a, b, c \ldots \text{constants})$$

This means that the displacements are not entirely determined by the stresses & strains.

④ The constants a & c represents a ~~transm~~ translatory motion of the body & the b is a small angle of rotation of the body

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(5) It has been shown that in the case of constant body forces the stress distribution is the same for plane stress distribution (or) plane strain.

(6) But the displacements are ~~difficult~~ different for these two problems, however, since in the case of plane stress distribution the components of strain

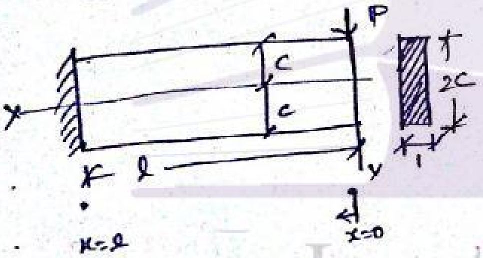
$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \quad \text{for plane stress}$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] = \frac{1}{E}\left[(1-\nu^2)\sigma_x - \nu(1+\nu)\sigma_y\right]$$

for plane strain

* Bending of a Cantilever Loaded at the End:-

(1) Consider a cantilever having a narrow rectangular c/s of unit width bent by a force "P" applied at the end.



(2) The upper & lower edges are free from load & shearing forces, having a resultant P, are distributed along the end $x=0$

(3) These condition can be satisfied by a proper combination of $\boxed{\text{pure shear with stresses}}$

(4) Superposing the pure shear $\tau_{xy} = -b_2$ on the stresses

$$(\tau_{xy})_{y=\pm c} = -b_2 - \frac{d_4}{2}c^2 = 0$$

$$d_4 = -2b_2/c^2$$

To satisfy the condition on the loaded end the sum of the shearing forces distributed over this end must be equal to P.

$$\int_{-c}^{c} \tau_{xy}\, dy = P \qquad \tau_{xy} = -b_2 - \frac{d_4}{2} c^2$$

$$\boxed{d_4 = -\frac{2b_2}{c^2}}$$

$$\int_{-c}^{c} \left(-b_2 - \frac{d_4}{2} y^2\right) dy = P$$

$$-\int_{-c}^{c} \left(b_2 + \frac{d_4}{2} y^2\right) dy = P \qquad -\int_{-c}^{c} \left(b_2 + \frac{\cancel{2}b_2}{c_2 \times \cancel{2}} y^2\right) dy = P$$

$$\cancel{-\int_{-c}^{c} b_2 - \frac{2b_2}{c^2 \times 2} \times c^2 = P} \qquad -\int_{-c}^{c} \left(b_2 - \frac{b_2}{c_2} y^2\right) dy = P$$

$$b_2 y - \frac{b_2}{c_2} \frac{y^3}{3} \Big|_{-c}^{c} = P$$

$$\boxed{\frac{1 \times 8c^3}{12}}$$

$$\left[b_2 c - \frac{b_2}{\cancel{c_2}} \frac{\cancel{c^3}}{3} - \left[-b_2 c + \frac{b_2}{\cancel{c_2}} \frac{\cancel{c^3}}{3}\right]\right] = +P$$

$$b_2 c - \frac{b_2 c}{3} + b_2 c - \frac{b_2 c}{3}$$

$$\frac{3 b_2 c - b_2 c + 3b_2 c - b_2 c}{3} = +P$$

$$\frac{4 b_2 c}{3} = +P \qquad \boxed{b_2 = \frac{3P}{4c}}$$

$$\sigma_x = d_4 xy = -\frac{2b_2}{c^2} x y$$

$$\sigma_x = -\frac{\cancel{2} \times 3P}{\cancel{4}_2 c \times c^2} = -\frac{3P}{2c^3} xy$$

$$\boxed{\sigma_x = \frac{-3P}{2c^3} xy} \qquad \sigma_y = 0 \qquad \tau_{xy} = \left(-b_2 - \frac{d_4}{2} y^2\right)$$

$$= \left(\frac{-3P}{4c} + \frac{\cancel{2} \times 3P}{\cancel{4}_2 c \times c^2 \times \cancel{2}} y^2\right)$$

$$\boxed{\tau_{xy} = \frac{-3P}{4c}\left(1 - \frac{y^2}{c^2}\right)}$$

Noting that $\frac{2}{3}c^3$ is the M.I (I) of cantilever c/s $1 \times 2c$

$$I = \frac{1 \times 2^3 \times c^3}{12} = \frac{8c^3}{12} = \frac{2}{3}c^3 \qquad \boxed{I = \frac{2}{3}c^3}$$

$$\sigma_x = \frac{-3P}{2c^3}xy = \frac{-Pxy}{I}$$

$$\boxed{\sigma_x = \frac{-Pxy}{I}} \qquad \sigma_y = 0$$

$$\tau_{xy} = \frac{-3P}{4c}\left(1 - \frac{y^2}{c^2}\right)$$

$$= -\frac{3P}{4c}\left(\frac{c^2 - y^2}{c^2}\right) \Rightarrow \frac{-3P}{4c^3}\left(c^2 - y^2\right)$$

$$\boxed{\tau_{xy} \Rightarrow -\frac{P}{2I}\left(c^2 - y^2\right)}$$

Applying hooke's law

$$\boxed{\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned}}$$

$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{-Pxy}{EI}$$

$$\varepsilon_y = -\nu\frac{\sigma_x}{E} = +\frac{\nu Pxy}{EI}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{-P}{2IG}\left(c^2 - y^2\right)$$

* the procedure for obtaining the components u & v by Integration

$$\int \frac{\partial u}{\partial x} = \int -\frac{Pxy}{EI} \qquad \boxed{u = \frac{-Px^2y}{2EI} + f(y)}$$

$$\int \frac{\partial v}{\partial y} = \int -\frac{\nu Pxy}{EI} \qquad \boxed{v = \frac{-\nu Pxy^2}{2EI} + f_1(x)}$$

Substitute u & v in γ_{xy} formula.

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\frac{Px^2}{2EI} + \frac{d\,f(y)}{dy} + \frac{\nu Py^2}{2EI} + \frac{df_1(x)}{dx} = \frac{-P}{2IG}\left(c^2 - y^2\right)$$

Denoting $F(x) = -\frac{Px^2}{2EI} + \frac{d f_1(x)}{dx}$

$$G(y) = \frac{df(y)}{dy} + \nu \frac{Py^2}{2EI} - \frac{Py^2}{2IG}$$

$$K = -\frac{Pc^2}{2IG}$$

then

$$F(x) + G(y) = K$$

Actually $F(x)$ must be some constant d (Means strain)

$G(y)$ " " " e

$$\underset{d+e}{\cancel{e+d}} = -\frac{Pc^2}{2IG}$$

$$d = -\frac{Px^2}{2EI} + \frac{d f_1(x)}{dx}$$

$$\frac{d f_1(x)}{dx} = d + \frac{Px^2}{2EI}.$$

$$e = \frac{d f(y)}{dy} + \frac{\nu Py^2}{2EI} - \frac{Py^2}{2IG}$$

$$\frac{d f(y)}{dy} = e + \frac{Py^2}{2IG} - \frac{\nu Py^2}{2EI}.$$

Functions $\boxed{f(y) = ey + \frac{Py^3}{6IG} - \nu\frac{Py^3}{6EI} + g}$

$$\boxed{f_1(x) = dx + \frac{Px^3}{6EI} + h}$$

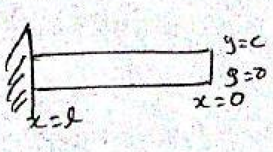
Substitute the Expression for u & v

$$u = -\frac{Px^2y}{2EI} + ey + \frac{Py^3}{6IG} - \frac{\nu Py^3}{6EI} + g$$

$$v = \frac{\nu Pxy^2}{2EI} + dx + \frac{Px^3}{6EI} + h$$

Boundary Conditions for finding d, e, g, h

@ $x=l, y=0$ $u=0$; $v=0$ then



then $g=0$; $h = -dl - \frac{Pl^3}{6EI}$

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The deflection curve is obtained by substituting $y=0$

$$V = \frac{\nu P x y^2}{2EI} + \frac{Px^3}{6EI} + dx + h$$

$$= \nu(0) + \frac{Px^3}{6EI} + dx + \frac{Pl^3}{6EI} - dl$$

$$\boxed{V = \frac{Px^3}{6EI} + \frac{Pl^3}{6EI} + (dx - dl)}$$

* For determine the constant d.

the B.C i.e., the rotation of beam in the xy plane about fixed point A.

$\frac{\partial v}{\partial x} = 0$ @ $x = l$ & $y = 0$ — The constrain can be realised in various ways

same as $\left(\frac{\partial u}{\partial y}\right) = 0$ @ $x = l$ & $y = 0$

$$d = -\frac{Pl^2}{2EI} \; ; \quad e = \frac{Pl^2}{2EI} - \frac{Pc^2}{2IG}$$

Substituting all elements in

$$v = -\frac{Px^2 y}{2EI} - \frac{\nu P y^3}{6EI} + \frac{Py^3}{6IG} + \left(\frac{Pl^2}{2EI} - \frac{Pc^2}{2IG}\right)y$$

$$V = \frac{\nu P x y^2}{2EI} + \frac{Px^3}{6EI} - \frac{Pl^2 x}{2EI} + \frac{Pl^3}{3EI}.$$

Substitute all value of @ $x=0$ & $y=0$

$$\boxed{V = Pl^3/3EI}$$

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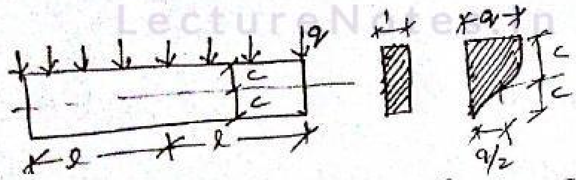
* **Bending of a Beam by Uniform Load:-**

① Let a beam of narrow rectangular cross section of unit width, supported at the ends, be bent by a uniformly distributed load of intensity q as.

② The upper & lower edge of the beam conditions are

$$(\tau_{xy})_{\pm c} = 0 \qquad (\sigma_y)_{+c} = 0 \qquad (\sigma_y)_{-c} = -q \rightarrow ①$$



③ ~~the conditions at the ends $x = \pm l$ are~~

$$\cancel{\int_{-c}^{c} \tau_{xy}\, dy =}$$

it state that there is no longitudinal force & no bending couple at the ends of the beam.

④ To remove the tensile stresses along the side $y = c$ & shearing stresses along $y = \pm c$ we super impose ~~the com~~ a simple compression $\sigma_y = a_2$ & the stresses $\sigma_y = b_3 y$ & $\tau_{xy} = -b_3 x$

$$\sigma_x = d_5 \left(x^2 y - \frac{2}{3} y^3\right)$$

$$\sigma_y = \frac{1}{3} d_5 y^3 + b_3 y + a_2$$

$$\tau_{xy} = -d_5 x y^2 - b_3 x$$

from ①

$$-d_5 c^2 - b_3 = 0$$

$$+\frac{1}{3} d_5 c^3 + b_3 c + a_2 = 0$$

$$-\frac{1}{3} d_5 c^3 - b_3 c + a_2 = -q$$

$$\boxed{a_2 = \frac{-q}{2}, \quad b_3 = \frac{3}{4}\frac{q}{c} \quad d_5 = \frac{-3}{4}\frac{q}{c^3}}$$

7

(1) ~~Develop the stiffness Matrix for the end loaded prismatic member AB with reference to the co-ordinates shown~~

Substituting $I = \frac{2c^3}{3}$

$$\sigma_x = -\frac{3}{4}\frac{q}{c^3}\left(x^2y - \frac{2}{3}y^3\right)$$

$$= -\frac{q}{2I}\left(x^2y - \frac{2}{3}y^3\right)$$

$$\sigma_y = \frac{1}{3}\left(\frac{-3}{4}\frac{q}{c^3}\right)y^3 + \frac{3}{4}\frac{q}{c}y - \frac{q}{2}$$

$$= \cancel{\frac{-q}{4c^3}y^3 + \frac{3}{4}\frac{q}{c} - \frac{q}{2}}$$

$$= \frac{-3}{4}\frac{q}{c^3}\left(\frac{1}{3}y^3 - c^2y + \frac{2}{3}c^3\right)$$

$$= \frac{-q}{2I}\left(\frac{1}{3}y^3 - c^2y + \frac{2}{3}c^3\right)$$

$$\tau_{xy} = +\frac{3q}{4c^3}y^2 - \frac{3q}{4c}$$

$$= -\frac{3q}{4c^3}\left(-c^2+y^2\right)x$$

To Make the couples at the ends of beam Vanish, we super Impose $\sigma_x = d_3 y$ i.e., $\sigma_y = \tau_{xy} = 0$

$$\int_{-c}^{c}\sigma_x \cdot y \cdot dy = \int_{-c}^{c}\left[d_5\left(x^2y - \frac{2}{3}y^3\right) + d_3 y\right]y \cdot dy = 0$$

from which $\boxed{d_3 = \frac{3}{4}\frac{q}{c}\left(\frac{l^2}{c^2} - \frac{2}{5}\right)}$

hence finally $\sigma_x = \frac{-3}{4}\frac{q}{c^3}\left(x^2y - \frac{2}{3}y^3\right) + \frac{3}{4}\frac{q}{c}\left(\frac{l^2}{c} - \frac{2}{5}\right)y$

$$\sigma_x = \frac{q}{2I}(l^2 - x^2)y + \frac{q}{2I}\left(\frac{2}{3}y^3 - \frac{2}{5}c^2 y\right)$$

$$\tau_{xy} = \frac{-q}{2I}(c^2 - y^2)x.$$

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\sigma_x}{E} :$$ $\rightarrow$ find u value with f(y) as function

$$\varepsilon_y = \frac{\partial v}{\partial y} = -\frac{\nu\sigma_x}{E}$$ $\rightarrow$ find v falue $f_1(x)$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\tau_{xy}}{G} \rightarrow (1)$$

Substitue u & v & in eq (1)

Separate the x & y terms

& given F(x) & G(y) for x & y terms

F(x) = x terms & take c = F(x)

& find the $f_1(x)$ & in terms of c with integrating constants ... - - Same procedure as 1^{st} problem

* **Solution of the Two-dimensional problem In the form of Fourier series:-**

(1) it has been shown that if the load is continuously distributed along the length of a rectangular beam of narrow section, a stress function in the form of polynomial may be used in certain simple cases.

(2) A much greater degree of generality is attained by taking the function as a fourier series.

The equation for the stress function

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$ may be satisfied by

taking the function

$$\boxed{\phi = \sin \frac{m\pi x}{l} f(y)} \rightarrow (1)$$

In which m is an Integer & f(y) a function of y only

$\frac{m\pi}{l} = \alpha$ then we find the following eqn for determining f(y):

$$\alpha^4 f(y) - 2\alpha^2 \cdot f''(y) + f^{IV}(y) = 0$$

constant co-efficients & the general integral of this linear D.f.e with constant co-efficients is

$$f(y) = C_1 \cosh \alpha y + C_2 \sinh \alpha y + C_3 y \cosh \alpha y + C_4 y \sinh \alpha y$$

from (1) the stress function

$$\boxed{\phi = \sin \alpha x \left[C_1 \cosh \alpha y + C_2 \sinh \alpha y + C_3 y \cosh \alpha y + C_4 y \sinh \alpha y\right]} \rightarrow (2)$$

the corresponding stress function

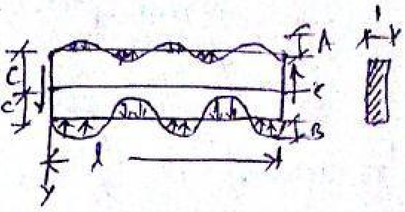
$$(3) \begin{cases} \sigma_x = \frac{\partial^2 \phi}{\partial y^2} = \sin \alpha x \left[C_1 \alpha^2 \cosh \alpha y + C_2 \alpha^2 \sinh \alpha y + C_3 \alpha (2 \sinh \alpha y + \alpha y \cosh \alpha y) + C_4 \alpha (2 \cosh \alpha y + \alpha y \sinh \alpha y)\right] \\ \sigma_y = \frac{\partial^2 \phi}{\partial x^2} = -\alpha^2 \sin \alpha x \left[C_1 \cosh \alpha y + C_2 \sinh \alpha y + C_3 y \cosh \alpha y + C_4 y \sinh \alpha y\right] \\ \tau_{xy} = \frac{-\partial^2 \phi}{\partial x \cdot \partial y} = -\alpha \cos \alpha x \left[C_1 \alpha \sinh \alpha y + C_2 \alpha \cosh \alpha y + C_3 (\cosh \alpha y + \alpha y \sinh \alpha y) + C_4 (\sinh \alpha y + \alpha y \cosh \alpha y)\right] \end{cases}$$

⇒ Let us consider a particular case of rectangular beam supported at the end and subjected to along upper & lower edges to the action of continuously distributed vertical forces of the intensity $A \sin \alpha x$ & $B \sin \alpha x$ respectively as here $\alpha = 4\pi/l$

⇒ for finding $C_1, C_2 \ldots C_4$ value the B.C are

for $y = +c$ $\quad \tau_{xy} = 0 \quad \sigma_y = -B \sin \alpha x$

for $y = -c$, $\quad \tau_{xy} = 0 \quad \sigma_y = -A \sin \alpha x$



⇒ Substituting these values in eq.ⁿ (e) of third eq.ⁿ then

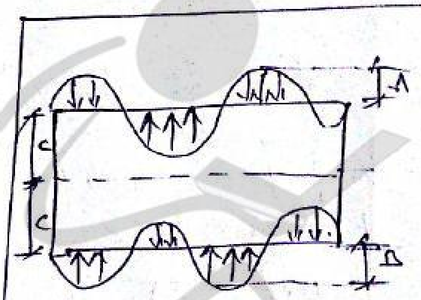
$$C_1 \alpha \sinh \alpha c + C_2 \alpha \cosh \alpha c + C_3 (\cosh \alpha c + \alpha c \sinh \alpha c) + C_4 (\sinh \alpha c + \alpha c \cosh \alpha c) = 0$$

$$-C_1 \alpha \sinh \alpha c + C_2 \alpha \cosh \alpha c + C_3 (\cosh \alpha c + \alpha c \sinh \alpha c) - C_4 (\sinh \alpha c + \alpha c \cosh \alpha c) = 0$$

from which

$$C_3 = -C_2 \frac{\alpha \cosh \alpha c}{\cosh \alpha c + \alpha c \sinh \alpha c}$$

$$C_4 = -C_1 \frac{\alpha \sinh \alpha c}{\sinh \alpha c + \alpha c \cosh \alpha c} \quad (4)$$



⇒ Substitute

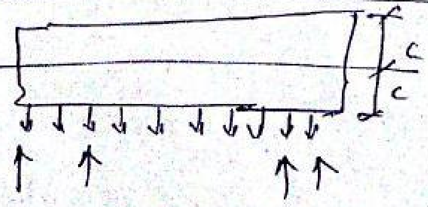
$$\sigma_y = B \ @ \ y = +c$$
$$\sigma_y = A \ @ \ y = -c$$

then

$$\alpha^2 (C_1 \cosh \alpha c + C_2 \sinh \alpha c + C_3 c \cosh \alpha c + C_4 c \sinh \alpha c) = B$$
$$\alpha^2 (C_1 \cosh \alpha c - C_2 \sinh \alpha c - C_3 c \cosh \alpha c + C_4 c \sinh \alpha c) = A \quad \to (5)$$

Substitute C_3 & C_4 values in eq.ⁿ (5), then we have 2 eq.ⁿ then we ~~5th eq.ⁿ will also~~ can find $c_1\ c_2\ c_3\ c_4$ and substitute in ~~eq.ⁿ~~ $\sigma_x\ \sigma_y\ \tau_{xy}$ enough

for gravity loading:-



in Gravity loading the self wt. of body acts as downward the B.C which we have

$\sigma_x = 0$; $\sigma_y = -\rho g (y+c)$

$\tau_{xy} = 0$

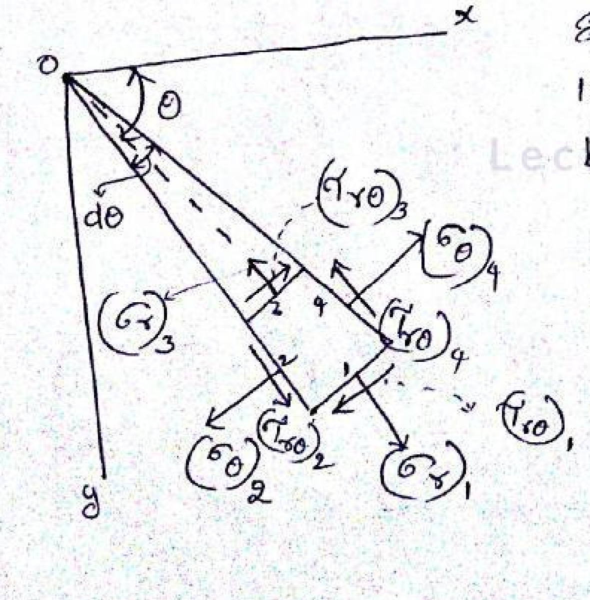
UNIT - III

Two dimensional problems in polar co-ordinates - General Equations in polar co-ordinates - stress distribution for problems having symmetrical about an axis - Strain components in polar co-ordinates - displacements for symmetrical stress distributions - stresses for plates with circular hole subjected to far field tension - stress concentration factor.

* General Equations in Polar Co-ordinates:

(1) In discussing stresses in circular rings & disks, curved bars of narrow rectangular cross section with a circular axis etc., it is advantageous to use polar co-ordinates.

(2) The position of a point in the middle plane of a plate is defined by the dist. from the origin O & by the angle "θ" b/w r & a certain axis Ox fixed in the plane.



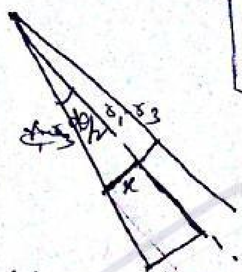
(3) Let us consider the Equilibrium of a small element 1234 cut out from the plate by the radial sections (4,2) normal to the plate & by two cylindrical surfaces 3,1 normal to the plate.

④ The radii of the sides of 3, 1 are denoted by r_3, r_1

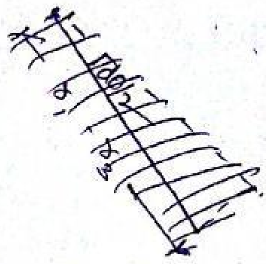
⑤ The radial force on the side 1 is $(\sigma_r)_1 r_1 d\theta$ which may be written as $(\sigma_r r)_1 d\theta$

⑥ & Similarly on side 3 is $-(\sigma_r r)_3 d\theta$

⑦ The normal force on side 2 has a component along the radius through P of $-(\sigma_\theta)_2 (r_1 - r_3) \sin(d\theta/2)$



$$\boxed{\sin\left(d\theta/2\right) = \frac{x}{r_1 - r_3}}$$



Which may be replaced by $-(\sigma_\theta)_2\, dr\, (d\theta/2)$

on side 4 is $-(\sigma_\theta)_4 \cdot dr\, (d\theta/2)$

⑧ The shearing forces on sides 2 & 4 give

$$\left[(\tau_{r\theta})_2 - (\tau_{r\theta})_4\right] dr$$

Summing up forces in the radial direction, including body force R per unit volume

$$(\sigma_r r)_1 d\theta - (\sigma_r r)_3 d\theta - (\sigma_\theta)_2 dr \frac{d\theta}{2} - (\sigma_\theta)_4 dr \frac{d\theta}{2} + \left[(\tau_{r\theta})_2 - (\tau_{r\theta})_4\right] dr + R r \cdot d\theta \cdot dr = 0$$

Dividing by $dr \cdot d\theta$

$$\frac{(\sigma_r r)_1 - (\sigma_r r)_3}{dr} - \frac{1}{2}\left[(\sigma_\theta)_2 + (\sigma_\theta)_4\right] + \frac{(\tau_{r\theta})_2 - (\tau_{r\theta})_4}{d\theta} + Rr = 0$$

if the dimensions of the element are now taken smaller & smaller, to the limit zero then

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0$$

- In tangential direction

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + S = 0$$

These equations take the place In equilibrium equations they are satisfied by putting

$$\boxed{\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}}$$

$$\boxed{\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}}$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r \cdot \partial \theta}$$

$$\boxed{\tau_{r\theta} = \frac{-\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)}$$

(9) We can consider the stress distribution In xy components σ_x, σ_y, τ_{xy}

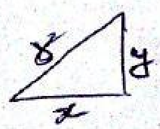
$$\sigma_r = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\sigma_\theta = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta$$

$$\tau_{r\theta} = (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

(10) We consider next the relations b/w derivatives In the two co-ordinate systems.

$$r^2 = x^2 + y^2 \qquad \theta = \frac{y}{x}$$



$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \qquad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$$

$\cancel{\partial x / \partial}\ \cancel{=\cos\theta}$

$$\frac{y}{r} = \sin\theta \qquad \frac{x}{r} = \cos\theta$$

$$\boxed{\frac{-y}{r^2} = -\frac{\sin\theta}{r}} \qquad \boxed{\frac{x}{r^2} = \frac{\cos\theta}{r}}$$

Thus for any function $f(x,y)$, In polar co-ordinates

$$\frac{\partial f}{\partial x} = \cos\theta \frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}$$

$$\frac{\partial^2 f}{\partial x^2} = \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta \cdot \frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}\right)$$

$$= \cos^2\theta \frac{\partial^2 f}{\partial r^2} - \cos\theta \cdot \sin\theta \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta \cdot \frac{\partial f}{\partial r}\right) + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\sin\theta \cdot \frac{\partial f}{\partial\theta}\right)$$

$$\frac{\partial^2 f}{\partial x^2} = \cos^2\theta \frac{\partial^2 f}{\partial r^2} + \cancel{\sin^2\theta\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right)} - \cos\theta \sin\theta \frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial f}{\partial\theta}\right)$$

$$- \cancel{\sin\theta \cos\theta \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right)}$$

$$- \sin\theta \cos\theta \frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial f}{\partial\theta}\right) + \sin^2\theta\left(\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{1}{r}\frac{\partial f}{\partial r}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = \sin^2\theta \frac{\partial^2 f}{\partial r^2} + \cos^2\theta\left(\frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}\right) + 2\sin\theta\cos\theta \frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial f}{\partial\theta}\right)$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial\theta^2}\right) f$$

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right)=0$$

From Various solutions of this partial Differential eqn we obtain solutions of two - dimensional problems in Polar co-ordinates for Various boundary conditions.

* Stress Distribution symmetrical About an axis:-

① When the stress function depends on r only, then the eqn of compatibility becomes

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}\right)=$$

$$\frac{\partial^4\phi}{\partial r^4}+\frac{\partial^3\phi}{\partial r^3}\times\frac{1}{r}+\frac{1}{r}\frac{\partial^3\phi}{\partial r^3}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial r^2}$$

$$\boxed{\frac{\partial^4\phi}{\partial r^4}+\frac{2}{r}\frac{\partial^3\phi}{\partial r^3}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial r^2}+\cancel{\frac{1}{r^3}\frac{\partial\phi}{\partial r}}=0}$$

② This is an ordinary differential eqn, which can be reduced to a linear diff. eqn with co-efficients by Introducing a new variable "t" Such that $\boxed{r=e^t}$

By substitution it can be checked that

$$\phi = A\log r + Br^2\log r + Cr^2 + D$$

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} = \frac{A}{r^2}+\frac{B}{r}\left(2r\log r+\frac{r^2}{r}\right)+\frac{C}{\cancel{r}}\times 2\cancel{r}$$

$$=\frac{A}{r^2}+B(2\log r+1)+2C$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} = \frac{-A}{r^2}+B(3+2\log r)+2C$$

$$\tau_{r\theta}=0$$

**
**

③ if there is no hole at the origin of co-ordinates, constants A & B vanish, since otherwise the stress components become infinite when $r=0$

④ Hence, for a plate without a hole at the origin & with no body forces $\sigma_r = \sigma_\theta$

taking for instance $B=0$

$$\sigma_r = \frac{A}{r^2} + 2C$$

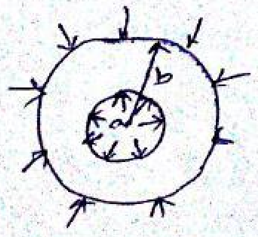
$$\sigma_\theta = -\frac{A}{r^2} + 2C$$

& the plate is in a condition of uniform tension (or) compression in all directions in its plane

⑤ This solution may be adopted to represent the stress distribution in a hollow cylinder subjected to uniform pressure on the inner & outer surfaces.

⑥ Let a & b denote the inner & outer radii of the cylinder & P_i & P_o the uniform internal & external pressures then the boundary conditions are

$(\sigma_r)_{r=a} = -P_i$ $\qquad$ $(\sigma_r)_{r=b} = -P_o$



⑦ Substituting we obtain

$$\frac{A}{a^2} + 2C = -P_i$$

$$\frac{A}{b^2} + 2C = -P_o$$

from which $A = \dfrac{a^2 b^2 (P_o - P_i)}{b^2 - a^2}$

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(4)

$$2C = \frac{P_i a^2 - P_o b^2}{b^2 - a^2}$$

then

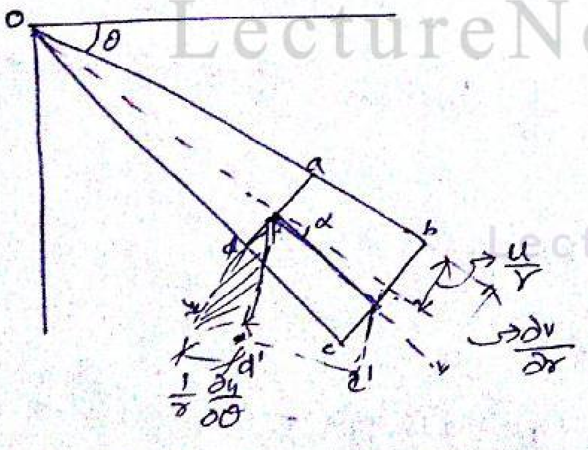
$$\sigma_r = \frac{a^2 b^2 (P_o - P_i)}{b^2 - a^2} \times \frac{1}{r^2} + \frac{P_i a^2 - P_o b^2}{b^2 - a^2}$$

$$\sigma_\theta = -\frac{a^2 b^2 (P_o - P_i)}{b^2 - a^2} \cdot \frac{1}{r^2} + \frac{P_i a^2 - P_o b^2}{b^2 - a^2}$$

* **Strain Components In Polar co-ordinates:-**

(1) In Considering the displacement In polar co-ordinates let us denote by u & v the components of displacement In the radial & tangential directions, respectively.

(2) if u is the radial displacement of the ~~ab~~ ad



the elongation In radial direction $\varepsilon_r = \frac{\partial u}{\partial r}$

(3) the strain In the tangential direction depends not only the displacement v but also on the radial displacement u.

(4) the tangential strain

$$e_\theta = \frac{u}{r} + \frac{\partial v}{r\partial\theta}$$

(5) Considering the shear strain, let α & β

~~the components~~

the shear strain $\gamma_{r\theta} = \frac{\partial u}{\partial\theta \cdot r} + \frac{\partial v}{\partial r} - \frac{v}{r}$

(6) the strain components interms of Hooke's law

$$e_r = \frac{1}{E}\left(\sigma_r - \nu\sigma_\theta\right)$$

$$e_\theta = \frac{1}{E}\left(\sigma_\theta - \nu\sigma_r\right)$$

$$\gamma_{r\theta} = \frac{1}{G}\tau_{r\theta}$$

---

* Displacements for Symmetrical Stress Distribution:-

$$\left.\begin{aligned}\sigma_r &= \frac{1}{r}\frac{\partial\phi}{\partial r} = \frac{A}{r^2} + B\left(1+2\log r\right) + 2C \\ \sigma_\theta &= \frac{\partial^2\phi}{\partial r^2} = -\frac{A}{r^2} + B\left(3+2\log r\right) + 2C\end{aligned}\right\} \rightarrow (1)$$

$$e_r = \frac{1}{E}\left(\sigma_r - \nu\sigma_\theta\right) = \frac{\partial u}{\partial r} \quad \text{Substitute } \sigma_r \text{ \& } \sigma_\theta \text{ values}$$

$$e_r = \frac{\partial u}{\partial r} = \frac{1}{E}\left[\frac{A}{r^2} + B + 2B\log r + 2C + \frac{\nu A}{r^2} - 3B\nu - 2B\nu\log r - \nu 2C\right]$$

$$\frac{\partial u}{\partial r} = \frac{1}{E}\left[\frac{A}{r^2}(1+\nu) + 2(1-\nu)B\log r + B(1-3\nu) + 2C(1-\nu)\right]$$

$$\cancel{u = \frac{1}{E}\left[\frac{-A}{r^2}(1+\nu)\right.}$$

$$\int 2(1-\nu)B\log r = \int (2B\log r - 2\nu B\log r)$$

$$= (2B\, r\log r - rB) - \int 2\nu B\log r$$

$$= (2B\, r\log r - rB) - [2\nu B\log r \times r - r]$$

$$= 2B\, r\log r - rB - 2\nu B r\log r + 2r\nu B$$

$$= \log r\, 2Br(1-\nu) - rB(1+2\nu)$$

$$\boxed{\int 2(1-\nu)B\log r = 2Br(1-\nu)\log r - B(1+2\nu)r}$$

$$u = \frac{1}{E}\left[-\frac{A}{r^2}(1+\nu) + 2Br(1-\nu)\log r - B(1+2\nu)r + (1-3\nu)Br + 2(1-\nu)Cr\right]$$

$$= \frac{1}{E}\left[-\frac{A}{r^2}(1+\nu) + 2Br(1-\nu)\log r - Br + 2\nu Br + Br - 3\nu Br + 2(1-\nu)Cr\right] + f(\theta)$$

$$\boxed{u = \frac{1}{E}\left[-\frac{A}{r^2}(1+\nu) + 2B(1-\nu)r\cdot\log r - B(1+\nu)r + 2C(1-\nu)r\right] + f(\theta)} \rightarrow (2)$$

$f(\theta)$ is a function of "θ" only

$$\left.\begin{aligned} \sigma_r &= \frac{1}{r}\frac{\partial\phi}{\partial r} = \frac{A}{r^2} + B(1+2\log r) + 2C \\ \sigma_\theta &= \frac{\partial^2\phi}{\partial r^2} = -\frac{A}{r^2} + B(3+2\log r) + 2C. \end{aligned}\right\} \rightarrow (3)$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$

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$$\varepsilon_\theta = \frac{1}{E}\left(\sigma_\theta - \nu\sigma_r\right) = \frac{u}{r} + \frac{\partial v}{r\partial\theta}$$

$$\cancel{\frac{\partial v}{\partial\theta}} = \frac{1}{E}\left[\frac{-A}{r^2} + 3B + 2B\log r + 2C - \nu\left(\frac{A}{r^2} + B + 2B\log r + 2C\right)\right]$$

$$= \frac{1}{E}\left[-\frac{A}{r^2} - \frac{\nu A}{r^2} + 3B - \nu B + 2B\log r - \nu\, 2B\log r + 2C - 2C\nu\right]$$

$$\cancel{\frac{1}{E}\left[-\frac{A}{r^2}(1+\nu) + B(3-\nu) + 2B\log r(1-\nu) + 2C(1-\nu)\right]}$$

$$\frac{\partial v}{r\partial\theta} = \frac{1}{E}\left[\frac{-A}{r^2} - \frac{\nu A}{r^2} + 3B - \nu B + 2B\log r - \nu\, 2B\log r + 2C - 2C\nu\right] - \frac{u}{r}$$

$$\frac{\partial v}{\partial\theta} = \left[\frac{1}{E}\left[\frac{-A}{r^2} - \frac{\nu A}{r^2} + 3B - \nu B + 2B\log r - \nu\, 2B\log r + 2C - 2C\nu\right] - \frac{u}{r}\right] r$$

$$= \frac{1}{E}\left[-\frac{A}{r} - \frac{\nu A}{r} + 3B\times r - \nu B r + 2B\log r\times r - \nu\, 2B\log r\times r + 2Cr - 2C\nu r\right] - \left[\frac{1}{E}\left(-\frac{(1+\nu)A}{r} + 2(1-\nu)Br\log r - B(1+\nu)r + 2C(1-\nu)r\right] - f(\theta)$$

$$= -\frac{A}{E}\left[\cancel{\frac{1+\nu}{r}}\right] + \frac{Br(3-\nu)}{E} + \cancel{\frac{2B\log r\times r(1-\nu)}{E}} + \cancel{\frac{2Cr(1-\nu)}{E}} + \cancel{\frac{(1+\nu)A}{Er}} - \cancel{\frac{2(1-\nu)Br\log r}{E}} + \frac{B(1+\nu)r}{E} - \cancel{\frac{2C(1-\nu)r}{E}} - f(\theta)$$

$$\frac{\partial v}{\partial\theta} = Br\frac{(1+\cancel{\nu}+3-\cancel{\nu})}{E} - f(\theta)$$

$$\frac{\partial v}{\partial\theta} = \frac{4Br}{E} - f(\theta)$$

$$\frac{\partial v}{\partial \theta} = \frac{4Br}{E} - f(\theta)$$

$$\boxed{v = \frac{4Br\theta}{E} - \int f(\theta)\, d\theta + f_1(r)} \rightarrow (4)$$

$f_1(r)$ is a function of r only

Substituting 2 Eq (4) in $\gamma_{r\theta} = \frac{\partial u}{r\,\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r}$ 4

$$\gamma_{r\theta} = \frac{\tau_{r\theta}}{G} = 0 \quad \left[\text{Since } \tau_{r\theta} = 0\right]$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial}{\partial \theta}\left[0 + f(\theta)\right] \; ; \; \frac{\partial v}{\partial r} = \frac{\partial}{\partial r}\left[\left(f_1(r)\right) + \frac{4Br\theta}{E}\right]$$

$$\frac{1}{r} \times \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = \frac{1}{r} \cdot \frac{\partial f(\theta)}{\partial \theta} + \frac{\partial f_1(r)}{\partial r} + \cancel{\frac{4B\theta}{E}} - \cancel{\frac{4Br\theta}{E}} \times \frac{1}{\cancel{r}} + \frac{\int f(\theta)\, d\theta}{r} - \frac{f_1(r)}{r} = 0$$

$$= \frac{1}{r} \frac{\partial f(\theta)}{\partial \theta} + \frac{\partial f_1(r)}{\partial r} + \frac{1}{r}\int f(\theta)\, d\theta - \frac{1}{r} f_1(r) = 0$$

From which $\boxed{f_1(r) = Fr}$ $\boxed{f(\theta) = H\sin\theta + K\cos\theta}$

Where F, H & K are constants to be determined from the conditions of constraint of the curved bar.

Substitute $f_1(r)$ & $f(\theta)$ values in (2) & (4)

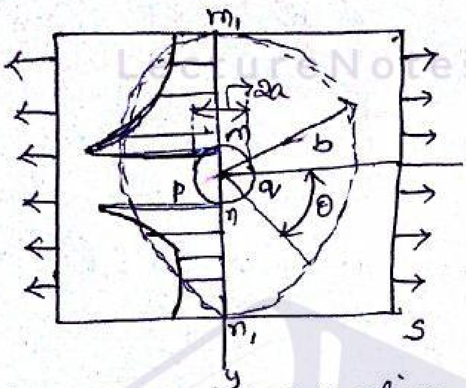
$$u = \frac{1}{E}\left[-\frac{(1+\nu)A}{r} + 2(1-\nu)Br\log r - B(1+\nu)r + 2C(1-\nu)r\right] + H\sin\theta + K\cos\theta$$

$$v = \frac{4Br\theta}{E} + Fr + H\cos\theta - K\sin\theta \rightarrow (5)$$

* In which the values of constants A, B & C for each particular case should be substituted.

~~* consider, pure bending~~

* The effect of circular holes on stress distribution in plates:-



① It shows a plate submitted to a uniform tension of magnitude 'S' in x direction.

② If a small circular hole is made in the middle of the plate, the stress distribution in the neighbourhood of the hole will be changed, but we can conclude from Saint-Venant's principle that change is negligible at distances which are large compared with a, the radius of the hole.

③ Consider the portion of the plate within a concentric circle of radius b, large in comparison with "a".

④ The stresses at the radius 'b' are effectively the same as in the plate without the hole &

$$(\sigma_r)_{r=b} = S\cos^2\theta = \frac{1}{2}S(1+\cos 2\theta)$$

$$(\tau_{r\theta})_{r=b} = -\frac{1}{2}S\sin 2\theta$$

these two stresses produces that may be derived from a stress function of the form.

$$\phi = f(r)\cos 2\theta$$

The general solution is

$$f(r) = Ar^2 + Br^4 + C \times \frac{1}{r^2} + D$$

$$\therefore \quad \phi = \left(Ar^2 + Br^4 + C\times\frac{1}{r^2} + D\right)\cos 2\theta$$

the corresponding stress components

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$= \frac{1}{r}\cos 2\theta \times \left[2Ar + 4Br^3 - \frac{2}{r^3}C\right] + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left[\left(Ar^2 + Br^4 + \frac{C}{r^2} + D\right)(-2\sin 2\theta)\right]$$

$$= \frac{1}{r}\cos 2\theta\left[2Ar + 4Br^3 - \frac{2C}{r^3}\right] + \frac{1}{r^2}\left[-4\left(Ar^2 + Br^4 + \frac{C}{r^2} + D\right)\cos 2\theta\right]$$

$$= \cos 2\theta\left[2A + \cancel{4Br^2} - \frac{2C}{r^4} - 4A - \cancel{Br^2 \times 4} - \frac{4C}{r^4} + \frac{4D}{r^2}\right]$$

$$= \cos 2\theta\left[-2A - \frac{6C}{r^4} - \frac{4D}{r^2}\right]$$

$$\boxed{\sigma_r = -\left(2A + \frac{6C}{r^4} + \frac{4D}{r^2}\right)\cos 2\theta} \rightarrow (1)$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} = \left(2A + 12Br^2 + \frac{6C}{r^4}\right)\cos 2\theta \rightarrow (3)$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = \left(2A + 6Br^2 - \frac{6C}{r^4} - \frac{2D}{r^2}\right)\sin 2\theta \rightarrow (2)$$

$$\cancel{2A + 12Br^2 + \frac{6C}{r^4}} \qquad 2A + \frac{6C}{b^4} + \frac{4D}{r^2}$$

if $r = b$; then $f(r) = \frac{1}{2}S$ form stress function $\phi = f(r)\cos 2\theta$

$$2A + \frac{6C}{b^4} + \frac{4D}{b^2} = -\frac{1}{2}S$$

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at the same time

$$2A + \frac{6C}{a^4} + \frac{4D}{a^2} = 0$$

in $\tau_{r\theta}$

$$\therefore 2A + 6Bb^2 - \frac{6C}{b^4} - \frac{2D}{b^2} = \frac{-1}{2}S$$

$$2A + 6Ba^2 - \frac{6C}{a^4} - \frac{2D}{a^2} = 0$$

from solving all eqn's

$A = -\frac{S}{4}$; $B = 0$; $C = -\frac{a^4}{4}S$; $D = \frac{a^2}{2}S$

Substitute all the A, B, C, D values in ①, ②, ③

then give σ_r, σ_θ, $\tau_{r\theta}$ values

UNIT - IV

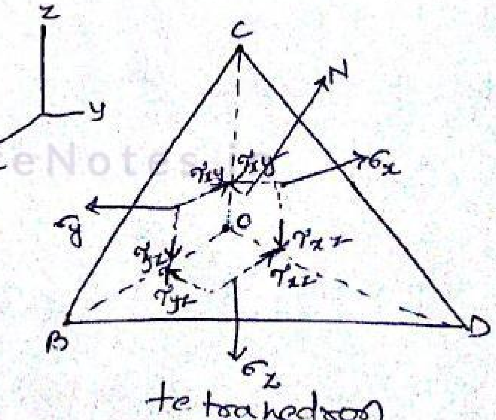
Analysis of Stress & strain In three dimension - Principal stresses - stress ellipsoid & stress director Surface - Determination of Principal Stresses - Max. Shear stress - Homogeneous Deformation - General theorems - Differential Equations of Equilibrium - Conditions of Compatibility - Equations of Equilibrium In terms of displacements - Principle of super position - Uniqueness of Solution - Reciprocal theorem.

* **Stress & strain In 3 dimension:-**

① It was shown that the stresses acting on the six sides of a cubic element can described by 6 components of stress 3 Normal components & 3 shearing stresses $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, $\tau_{yz} = \tau_{zy}$

② if these components of stress at any point are known, the stress acting on any Inclined plane through this point can be calculated from the equations of statics.

③ To get the stress for any Inclined plane through O, we can take a plane BCD parallel to it at a small distance from O



tetrahedron

④ Since the stresses vary continuously over the volume of the body

⑤ body forces are neglected, The forces acting on the tetrahedron can therefore be det. by multiplying the stress components by the areas of the faces.

(6) if A denotes the area of the face BCD of the tetrahedron if N is the Normal to the plane BCD

$$\cos(Nx) = l \qquad \cos(Ny) = m \qquad \cos(Nz) = n$$

the Areas of the three other faces of

$$Al \qquad Am \qquad An$$

(7) if we denote X, Y, Z the 3 components of stress, Also the components of forces in the x-direction acting on the 3 other faces

$$-Al\sigma_x \quad , \quad -Am\tau_{xy} \quad , \quad -An\tau_{xz}$$

the corresponding eqn of the tetrahedron is

$$AX - Al\sigma_x - Am\tau_{xy} - An\tau_{xz} = 0$$

$$\boxed{X = l\sigma_x + m\tau_{xy} + n\tau_{xz}} \rightarrow (1)$$

$$Y = \tau_{xy}l + m\sigma_y + n\tau_{yz} \rightarrow (2)$$

$$Z = \tau_{zx}l + m\tau_{zy} + n\sigma_z \rightarrow (3)$$

* Principal stresses:-

(1) Let us now consider the normal components of stress σ_n, acting on the plane

$$\sigma_n = Xl + Ym + Zn \rightarrow (4)$$

Substitute all X, Y, Z values in eqn (4)

$$\sigma_n = \sigma_x l^2 + m\tau_{xy}l + n\tau_{yz}l + \tau_{xy}lm + m\sigma_y m + n\tau_{yz}m + \tau_{zx}ln + m\tau_{zy}n + n\sigma_z n$$

$$\boxed{\sigma_n = \sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2\tau_{xy}lm + 2\tau_{yz}mn + 2\tau_{zx}nl} \rightarrow (4)$$

② The variation of σ_n with the direction of the Normal "N" can be represented

Let us

put in the direction of N a vector whose length "r" is inversely proportional to square root of the absolute value of the stress σ_n

$$r = \frac{K}{\sqrt{|\sigma_n|}}$$ K = constant factor

③ the co-ordinates of the end of this vector will be

$x = lr$; $y = mr$; $z = nr$

$$\sqrt{\sigma_n} = \frac{K}{r}$$

$$\sigma_n = \pm\frac{K^2}{r^2}$$

$$\pm K^2 = \sigma_n r^2$$

from eqn ④

$$\pm K^2 = \left[\sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2\tau_{xy} lm + 2\tau_{yz} mn + 2\tau_{zx} nl\right] r^2$$

$$= \sigma_x l^2 r^2 + \sigma_y m^2 r^2 + \sigma_z n^2 r^2 + 2\tau_{xy} lm r^2 + 2\tau_{yz} mn r^2 + 2\tau_{zx} nl r^2$$

$$\boxed{\pm K^2 = \sigma_x x^2 + \sigma_y y^2 + \sigma_z z^2 + 2\tau_{xy} xy + 2\tau_{yz} yz + 2\tau_{zx} zx} \rightarrow ⑤$$

④ As the plane BCD rotates about the point O, then end of the vector 'r' always lies on the surface of the second degree (Eqn 5)

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⑤ It is well known that in the case of a surface of the second degree (equ ⑤), it is always possible to find for the axes x, y, z such directions that the terms in this equation containing the products of co-ordinates vanish.

⑥ This means that we can always find 3 perpendicular planes for which $\tau_{yz}, \tau_{zx}, \tau_{xy}$ vanish that, the resultant stresses are perpendicular to the planes on which they act.

⑦ We call these stresses "principal stresses", their directions "principal axes" & the planes on which they act "principal planes"

---

* stress Ellipsoid & stress - director surface:-

① if the co-ordinate axes x, y, z are taken in the directions of the principal axes, calculation of the stress on any inclined plane becomes very simple

② The shearing stresses $\tau_{yz}, \tau_{zx}, \tau_{xy}$ are zero in this case

$$X = \sigma_x l \; ; \quad Y = \sigma_y m \; ; \quad Z = \sigma_z n$$

take $l^2+m^2+n^2=1$

$$\boxed{\frac{X^2}{\sigma_x^2}+\frac{Y^2}{\sigma_y^2}+\frac{Z^2}{\sigma_z^2}=1} \rightarrow ⑥$$

③ This means that, if for each inclined plane through a point "O" the stress is represented by a vector from "O" with the components X, Y, Z, the ends of all such vectors lie on the surface of the ellipsoid. this ellipsoid is called the "stress ellipsoid".

④ From this it can be concluded that the max. stress at any point is the largest of the three principal stresses at this point.

⑤ if two of the three principal stresses are numerically equal, the stress ellipsoid becomes an ellipsoid of revolution.

⑥ if these numerically equal principle stresses are of the same sign, the resultant stresses at each point on all planes through the axis of symmetry of the ellipsoid will be equal & perpendicular to the planes on which they act.

⑦ In this case, the stresses on any two perpendicular planes through this axis can be considered as principal stresses.

⑧ if all three principal stresses are equal & of the same sign, the stress ellipsoid becomes a sphere and any three perpendicular directions can be taken as "principal axes".

⑨ When one of the principal stresses is zero, the stress ellipsoid reduces the area of an ellipse & the vectors representing the stresses on all the planes through the point lies in the same plane.

⑩ this condition of stress is called "plane stress".

~~⑪ Each radius vector of the stress ellipsoid represents to a certain scale, the stress on one of the plane through the center of the ellipsoid.~~

⑪ the "stress - director" surface defined by the eqn

$$\frac{x^2}{\sigma_x} + \frac{y^2}{\sigma_y} + \frac{z^2}{\sigma_z} = 1$$

(12) the Stress represented by a radius vector of the stress ellipsoid acts on the plane parallel to the tangent plane to the stress-director at the point of its inter-section with the radius vector.

(13) The equation of the tangent plane to the stress-director surface at any point x_0, y_0, z_0 is

$$\boxed{\frac{x x_0}{\sigma_x} + \frac{y y_0}{\sigma_y} + \frac{z z_0}{\sigma_z} = 1} \rightarrow (7)$$

~~(14) Denoting by h leng~~

(14) Denoting by "h" the Length of the perpendicular from the origin of co-ordinates to the above tangent plane & by l, m, n the direction cosines of this perpendicular, the eqn of this tangent plane can be written as

$$\boxed{lx + my + nz = h} \rightarrow (8)$$

& $\sigma_x = \frac{x_0 h}{l}$; $\sigma_y = \frac{y_0 h}{m}$; $\sigma_z = \frac{z_0 h}{n}$

* Determination of the principal stresses:-

(1) if the stress components for three co-ordinate planes are known, we can det= the directions & magnitudes of the principal stresses by using this property that the principal stresses are perpendicular to the planes on which they act.

(2) Let l, m, n be the direction cosines of a principal plane & "S" the magnitude of the principal stress acting on this plane. then the components of this stress are

$$X = Sl \qquad Y = Sm \qquad Z = Sn$$

Substitue In eqn (1) (2) (3)

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(4)

$$(S-\sigma_x)l - \tau_{xy} m - \tau_{xz} n = 0$$

$$-\tau_{xy} l + (S-\sigma_y) m - \tau_{yz} n = 0$$

$$-\tau_{xz} l - \tau_{yz} m + (S-\sigma_z) n = 0$$

(3) These are three homogeneous linear equations in l, m, n. They will give solutions diff. from zero only if the determinant of these eqn's is zero

Calculating this determinant & putting it equal to zero give the following cubic eqn in "S"

$$\boxed{S^3 - (\sigma_x+\sigma_y+\sigma_z) S^2 + (\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{yz}^2 - \tau_{zx}^2 - \tau_{xy}^2) S - (\sigma_x\sigma_y\sigma_z + 2\tau_{yz}\tau_{xz}\tau_{xy} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2) = 0} \rightarrow (9)$$

(4) the 3 roots of this eqn give the values of the 3 principal stresses S_1, S_2, S_3.

(5) By substituting each of these stresses in eqn $\cos(Nx) = l$; $\cos(Ny) = m$ $\cos(Nz) = n$

Using the relation $l^2+m^2+n^2=1$, we can find three sets of direction cosines for the three principal planes.

* Stress Invariants:-

(1) Regarding the state of stress, i.e, the Principal stresses & Principal axes, as given, we can of course represent it by components in any set of x, y, z axes.

(2) No matter what orientation is chosen for these axes, eqn (9) must given the same three roots for 'S'.

(3) Consequently, the coefficients must always be the same.

i.e, $$\boxed{\sigma_x + \sigma_y + \sigma_z = S_1 + S_2 + S_3} \rightarrow 10$$

$$\boxed{\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_x \sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 = S_1 S_2 + S_2 S_3 + S_3 S_1} \rightarrow (11)$$

$$\boxed{\sigma_x \sigma_y \sigma_z + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_x \tau_{yz}^2 - \sigma_y \tau_{xz}^2 - \sigma_z \tau_{xy}^2 = S_1 S_2 S_3} \rightarrow (12)$$

(4) The Expressions on the left are "stress Invariants."

(5) Evidently other Invariant Expressions can be formed from them.

i.e, $$\boxed{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2) = 2I_1^2 - 6I_2} \rightarrow (13)$$

* Determination of the Max. Shearing Stress:-

(1) Let x, y, z be the principal axes so that $\sigma_x, \sigma_y, \sigma_z$ are Principal stresses & let l, m, n be the direction cosines for a given plane.

Then the square of the total stress on this plane is

$$\boxed{S^2 = X^2 + Y^2 + Z^2 = \sigma_x^2 l^2 + \sigma_y^2 m^2 + \sigma_z^2 n^2} \rightarrow (14)$$

(2) The square of the Normal component of the stress

$$\sigma_n^2 = (\sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2)^2$$

Then the square of the shearing stress on the same plane

(3) $\tau^2 = S^2 - \sigma_n^2$

$$= \sigma_x^2 l^2 + \sigma_y^2 m^2 + \sigma_z^2 n^2 - (\sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2)^2 \rightarrow (15)$$

We shall eliminating one of the direction cosines cosines, say n, using the relation

$$\boxed{l^2 + m^2 + n^2 = 1}$$

$$n^2 = 1 - l^2 - m^2$$

(4) derivation of (15) eqn with respect to l)

$$\frac{d}{dl}\left(\sigma_x^2 l^2 + \sigma_y^2 m^2 + \sigma_z^2 n^2 - (\sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2)^2\right)$$

and equating to its zero, we obtain the following equations for determining direction cosines of the planes for which τ is a max. (or) minimum.

$$\cancel{2\sigma_x^2 l + 2(\sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2)(2\sigma_x l) = 0}$$

$$\frac{d}{dl}\left(\sigma_x^2 l^2 + \sigma_y^2 m^2 + \sigma_z^2 (1 - l^2 - m^2) - (\sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2)^2\right)$$

$$\frac{d}{dl}\left(\sigma_x^2 l^2 + \sigma_y^2 m^2 + \sigma_z^2 - \sigma_z^2 l^2 - m^2 \sigma_z^2 - (\sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2)^2\right)$$

$$\left[2\sigma_x^2 l - 2\sigma_z^2 l - 2(\sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2)(2\sigma_x l)\right] = 0$$

$$\left[2\sigma_x^2 l - 2\sigma_z^2 l - (2\sigma_x l^2 - 2\sigma_y m^2 - 2\sigma_z n^2)2\sigma_x l\right] = 0$$

$$\left[2\sigma_x^2 l - 2\sigma_z^2 l - 4\sigma_x^2 l^3 + 4\sigma_x \sigma_y m^2 l + 2\sigma_z \sigma_x n^2 l\right] = 0$$

$$l\left[2\sigma_x^2 - 2\sigma_z^2 - 4\sigma_x^2 l^2 + 4\sigma_x \sigma_y m^2 + 2\sigma_z \sigma_x n^2\right] = 0$$

$$l\left[(\sigma_x - \sigma_z)l^2 + (\sigma_y - \sigma_z)m^2 - \frac{1}{2}(\sigma_x - \sigma_z)\right] = 0$$

$$m\left[(\sigma_x - \sigma_z)l^2 + (\sigma_y - \sigma_z)m^2 - \frac{1}{2}(\sigma_y - \sigma_z)\right] = 0 \qquad \rightarrow (16)$$

→ One solution of these eq^n obtained by putting $l = m = 0$.

→ Taking for instance $l = 0$; $m = \pm\sqrt{1/2}$

Suppose $m = 0$ then $l = \pm\sqrt{1/2}$

There are no solution for (16) eq^n. in which l & m are both diff. from zero

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→ Repeating the above calculations by eliminating from Expression "τ^2" the following table of direction cosines making τ a max. (or) minimum.

l	0	0	±1	0	$\pm\sqrt{1/2}$	$\pm\sqrt{1/2}$
m	0	±1	0	$\pm\sqrt{1/2}$	0	$\pm\sqrt{1/2}$
n	±1	0	0	$\pm\sqrt{1/2}$	$\pm\sqrt{1/2}$	0

→ The first three columns give the direction of the planes of co-ordinates, coinciding, as was assumed originally with the principal planes. For these planes the shearing stress is zero i.e., τ is a minimum.

→ The three remaining substitute the direction cosines

$$\tau = \pm\frac{1}{2}(\sigma_y - \sigma_z) \qquad \tau = \pm\frac{1}{2}(\sigma_x - \sigma_y)$$

$$\tau = \pm\frac{1}{2}(\sigma_x - \sigma_z)$$

This shows that the max. shearing stress acts on the plane bisecting the angle b/w the largest & the smallest principal stress ~~& is equal to half the difference of these two princi~~

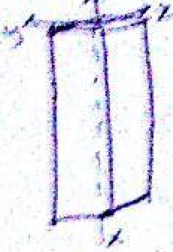
* Homogeneous Deformation:-

① We consider only small deformations such as occur in Engg. structures.

② The small displacements of the particles of a deformed body will usually be resolved into components u, v, w parallel to the co-ordinate axes x, y, z respectively.

(2) It will be assumed that these components are very small quantities varying continuously over the volume of the body.

(4) Consider, as an example, simple tension of a prismatical bar fixed at the upper end.



(5) Let ϵ be the unit elongation of the bar in the x direction & $\nu\epsilon$ the unit lateral contraction.

(6) Then the components of displacement of a point with co-ordinates x, y, z are

$u = \epsilon x$ $\quad v = -\nu\epsilon y$ $\quad w = -\nu\epsilon z$

(7) Denoting by x', y', z' the co-ordinates of the point after deformation

$$\boxed{\begin{aligned} x' &= x + u = x(1+\epsilon) \\ y' &= y(1-\nu\epsilon) \\ z' &= z(1-\nu\epsilon) \end{aligned}} \rightarrow (17)$$

(8) If we consider a plane in the bar before deformation such as that given by $ax + by + cz + d = 0$ → (18)

(9) The points of this plane will still be in a plane after deformation. The eqn of this new plane is obtained

(10) by substituting in eqn (18) the value of x, y, z from eqn (17)

(11) If we consider a spherical surface in the bar before deformation such as given by the eqn

$$\boxed{x^2 + y^2 + z^2 = r^2}$$

(11) This sphere becomes an ellipsoid after deformation, the eqn of which can be found by substituting in

$$x^2+y^2+z^2=r^2$$

$$\frac{x^2+y^2+z^2}{r^2}=\frac{r^2}{r} \Rightarrow \frac{x^2}{r^2}+\frac{y^2}{r^2}+\frac{z^2}{r^2}=1$$

after deformation $\frac{x'^2}{r^2(1+\epsilon)^2}+\frac{y'^2}{r^2(1-\nu\epsilon)^2}+\frac{z'^2}{r^2(1-\nu\epsilon)^2}=1$

Thus a sphere of radius r deforms into an ellipsoid with semi-axes $r(1+\epsilon)$, $r(1-\nu\epsilon)$, $r(1-\nu\epsilon)$

(12) The simple extension & lateral contraction, considered above. proceeding as before, it can be shown that this type of deformation has all the properties found above for the case of simple tension

(13) planes and straight lines remain plane & straight after deformation.

(14) A sphere becomes, after deformation, an ellipsoid.

(15) This kind of deformation is called "homogeneous deformation".

(16) It will be shown later that in this case the deformation in any given direction is the same at all the points of the deformed body.

(17) Thus two geometrically similar & similarly oriented elements of the body remain geometrically similar after distortion.

* Differential Equations of Equilibrium:-(3D)

(1) Same as 2-D but extra 3-D face is include

the final D.E. of E:-

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0 \\ \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + Z = 0 \end{aligned}\right\} \rightarrow 18$$

(2) these equations must satisfied at all points throughout the volume of the body.

(3) The stresses vary over the volume of the body. & when we arrive at the surface they must be such as to be in equilibrium with the External forces on the surface of the body.

(4) these conditions of equilibrium at the surface can be obtained from eq. (1), (2), (3)

$$\boxed{\begin{aligned} \bar{X} &= \sigma_x l + \tau_{xy} m + \tau_{xz} n \\ \bar{Y} &= \sigma_y m + \tau_{yz} n + \tau_{xy} l \\ \bar{Z} &= \sigma_z n + \tau_{xz} l + \tau_{yz} m \end{aligned}} \rightarrow \text{these are surface forces/unit area at boundary}$$

(5) if the problem is to determine the state of stress in a body submitted to the action of given forces it is necessary to solve eq. (18) we have 3 boundary conditions having (6) unknown values.

(6) So they are not sufficient for the determination of these components. the additional conditions called "Conditions of compatability".

* Conditions of Compatibility:-

① It should be noted that the six components of strain at each point are completely determined by 3 functions u, v, w representing the components of displacements.

② from 1st unit Eqn ②

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{\partial^3 u}{\partial x \cdot \partial y^2} \qquad \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^3 v}{\partial y \cdot \partial x^2} \qquad \frac{\partial^2 \gamma_{xy}}{\partial x \cdot \partial y} = \frac{\partial^3 u}{\partial x \cdot \partial y} + \frac{\partial^3 v}{\partial y \partial x^2}$$

from which

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \cdot \partial y} \longrightarrow (19)$$

③ two more relations of the same kind can be obtained by

$$\frac{\partial^2 \varepsilon_x}{\partial y \cdot \partial z} = \frac{\partial^3 u}{\partial x \partial y \partial z} \qquad \frac{\partial \gamma_{yz}}{\partial x} = \frac{\partial^2 v}{\partial z \cdot \partial x} + \frac{\partial^2 w}{\partial y \cdot \partial x} \rightarrow (20)$$

$$\frac{\partial \gamma_{xz}}{\partial y} = \frac{\partial^2 u}{\partial z \cdot \partial y} + \frac{\partial^2 w}{\partial x \partial y} \rightarrow (21) \qquad \frac{\partial \gamma_{xy}}{\partial z} = \frac{\partial^2 u}{\partial y \cdot \partial z} + \frac{\partial^2 v}{\partial z \cdot \partial x} \rightarrow (22)$$

we find that

$$\cancel{2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}}$$

Suppose

$$2 \cdot \frac{\partial^2 \varepsilon_x}{\partial y \cdot \partial z} = \frac{\partial^3 u}{\partial x \cdot \partial y \cdot \partial z} + \frac{\partial^3 u}{\partial x \cdot \partial y \cdot \partial z}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \cdot \partial z} + \frac{\partial^2 u}{\partial y \cdot \partial z}\right) \quad \text{from (21) \& (22)}$$

$$\frac{\partial^2 u}{\partial z \cdot \partial y} = +\frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial^2 w}{\partial x \cdot \partial y} \quad ; \quad \frac{\partial^2 u}{\partial y \cdot \partial z} = +\frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial^2 v}{\partial z \cdot \partial x}$$

then

$$2\frac{\partial^2 \varepsilon_x}{\partial y \cdot \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

Same as two relations

$$\left\{\begin{aligned} \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} &= \frac{\partial^2 \gamma_{xy}}{\partial x \cdot \partial y} \\ \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} &= \frac{\partial^2 \gamma_{yz}}{\partial y \cdot \partial z} \\ \frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} &= \frac{\partial^2 \gamma_{xz}}{\partial x \cdot \partial z} \end{aligned}\right. \quad (23)$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y \cdot \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_y}{\partial x \cdot \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \cdot \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

These differential relations are called "conditions of compatibility".

* By using hooke's law Eqn (23) can be transferred into stress components

take

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \cdot \partial z} \rightarrow (C)$$

In first chapter using

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu(\sigma_x + \sigma_z)\right)$$

$$\theta = \sigma_x + \sigma_y + \sigma_z$$

$$\sigma_z = \theta - \sigma_x - \sigma_y$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu(\cancel{\sigma_x} + \theta - \cancel{\sigma_x} - \sigma_y)\right)$$

$$\boxed{\varepsilon_y = \frac{1}{E}\left((1+\nu)\sigma_y - \nu\theta\right)}$$

$$\boxed{\varepsilon_z = \frac{1}{E}\left[(1+\nu)\sigma_z - \nu\theta\right]}$$

$$\gamma_{yz} = \frac{1}{G}\tau_{yz} \qquad G = \frac{E}{2(1+\nu)}$$

$$\boxed{\gamma_{yz} = \frac{2(1+\nu)}{E}\tau_{yz}}$$

Substituting these Expressions in (c)

$$\frac{\partial^2}{\partial z^2}\left(\frac{(1+\nu)\sigma_y - \nu\theta}{E}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{(1+\nu)\sigma_z - \nu\theta}{E}\right) = \frac{\partial^2}{\partial y\,\partial z}\left[\frac{2(1+\nu)\tau_{yz}}{E}\right]$$

$$\boxed{(1+\nu)\left[\frac{\partial^2\sigma_y}{\partial z^2} + \frac{\partial^2\sigma_z}{\partial y^2}\right] - \nu\left[\frac{\partial^2\theta}{\partial z^2} + \frac{\partial^2\theta}{\partial y^2}\right] = 2(1+\nu)\frac{\partial^2\tau_{yz}}{\partial y\,\partial z}} \rightarrow c'$$

From eqn (18)

$$\left.\begin{aligned} \frac{\partial\tau_{yz}}{\partial y} &= -\frac{\partial\sigma_z}{\partial z} - \frac{\partial\tau_{xz}}{\partial x} - Z \\ \frac{\partial\tau_{yz}}{\partial z} &= -\frac{\partial\sigma_y}{\partial y} - \frac{\partial\tau_{xy}}{\partial x} - Y \end{aligned}\right\} \begin{matrix}\text{first}\\ \text{second}\end{matrix}$$

differentiating the first w.r.t z

$$\frac{\partial^2\tau_{yz}}{\partial y^2} = -\frac{\partial^2\sigma_z}{\partial z^2} - \frac{\partial^2\tau_{xz}}{\partial x\,\partial z} - \frac{\partial Z}{\partial z}$$

differentiating the second w.r.t y

$$\frac{\partial^2\tau_{yz}}{\partial y^2} = -\frac{\partial^2\sigma_y}{\partial y^2} - \frac{\partial^2\tau_{xy}}{\partial x\,\partial y} - \frac{\partial Y}{\partial y}$$

adding two (2) eqns.

$$\frac{2\,\partial^2\tau_{yz}}{\partial y^2} = -\frac{\partial^2\sigma_z}{\partial z^2} - \frac{\partial^2\sigma_y}{\partial y^2} - \frac{\partial}{\partial x}\left(\frac{\partial\tau_{xz}}{\partial z} + \frac{\partial\tau_{xy}}{\partial y}\right) - \frac{\partial Z}{\partial z} - \frac{\partial Y}{\partial y} \rightarrow (24)$$

In eqn (18) the first eqn.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0$$

diff-e these eqn w.r.t "x"

$$\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \tau_{xy}}{\partial x \cdot \partial y} + \frac{\partial^2 \tau_{xz}}{\partial z \cdot \partial x} + \frac{\partial X}{\partial x} = 0$$

So

$$-\frac{\partial^2 \tau_{xy}}{\partial x \cdot \partial y} + \frac{\partial^2 \tau_{xz}}{\partial z \cdot \partial x} = +\frac{\partial X}{\partial x} + \frac{\partial^2 \sigma_x}{\partial x^2} \rightarrow (d)$$

Substitute "d" value in eqn (24)

$$\boxed{\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial^2 \sigma_z}{\partial z^2} + \frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z} = 2\partial^2 \tau_{yz}/\partial y \cdot \partial z} \rightarrow (25)$$

Substitute eqn (25) in c'

$$(1+\nu)\left(\frac{\partial^2 \sigma_y}{\partial z^2} + \frac{\partial^2 \sigma_z}{\partial y^2}\right) - \nu\left(\frac{\partial^2 \theta}{\partial z^2} + \frac{\partial^2 \theta}{\partial y^2}\right) = 2(1+\nu)\frac{\partial^2 \tau_{yz}}{\partial y \cdot \partial z}$$

$$(1+\nu)\left(\frac{\partial^2 \sigma_y}{\partial z^2} + \frac{\partial^2 \sigma_z}{\partial y^2}\right) - \nu\left(\frac{\partial^2 \theta}{\partial z^2} + \frac{\partial^2 \theta}{\partial y^2}\right) = (1+\nu)\left[\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial^2 \sigma_z}{\partial z^2} + \frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right]$$

$$\boxed{(1+\nu)\left(\frac{\partial^2 \sigma_y}{\partial z^2} + \frac{\partial^2 \sigma_z}{\partial y^2}\right) + (1+\nu)\left[\frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_z}{\partial z^2}\right] - (1+\nu)\left[\frac{\partial^2 \sigma_x}{\partial x^2}\right] - \nu\left(\frac{\partial^2 \theta}{\partial z^2} + \frac{\partial^2 \theta}{\partial y^2}\right) = (1+\nu)\left[\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right]} \rightarrow (26)$$

actually $\theta = \sigma_x + \sigma_y + \sigma_z$

Suppose take $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$\boxed{\nabla^2 \theta = \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2}} \rightarrow (27)$$

from eq^n (27)
$$\boxed{\nabla^2\theta - \frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}} \rightarrow (28)$$

~~Substitute in eq^n (26)~~

Same as in eq^n (26)

$$(1+\nu)\left(\frac{\partial^2\sigma_y}{\partial z^2}+\frac{\partial^2\sigma_z}{\partial y^2}\right) + (1+\nu)\left(\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial^2\sigma_z}{\partial z^2}\right) - (1+\nu)\frac{\partial^2\sigma_x}{\partial x^2} - \nu\left(\frac{\partial^2\theta}{\partial z^2}+\frac{\partial^2\theta}{\partial y^2}\right) =$$
$$(1+\nu)\left(\frac{\partial X}{\partial x}-\frac{\partial Y}{\partial y}-\frac{\partial Z}{\partial z}\right) \rightarrow (29)$$

~~$\nabla^2\theta$~~ actually ~~$\nabla^2\theta = \frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}$~~

$$\boxed{\nabla^2\theta - \frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}} \rightarrow 30$$

also
$$\boxed{\nabla^2\sigma_x = \frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_x}{\partial y^2}+\frac{\partial^2\sigma_x}{\partial z^2}} \rightarrow 31$$

from 30 & 31

$$\boxed{\nabla^2\theta - \frac{\partial^2\theta}{\partial x^2} - \nabla^2\sigma_x = \frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2} - \frac{\partial^2\sigma_x}{\partial x^2} - \frac{\partial^2\sigma_x}{\partial y^2} - \frac{\partial^2\sigma_x}{\partial z^2}} \rightarrow 32$$

$= \cancel{\partial^2}$

where $\theta = \sigma_x+\sigma_y+\sigma_z$ then

Eq^n (32) becomes

$$= \cancel{\frac{\partial^2}{\partial y^2}\sigma_x} + \frac{\partial^2}{\partial y^2}\sigma_y + \frac{\partial^2}{\partial y^2}\sigma_z + \cancel{\frac{\partial^2}{\partial z^2}\sigma_x} + \frac{\partial^2\sigma_y}{\partial z^2} + \frac{\partial^2\sigma_z}{\partial z^2} - \frac{\partial^2\sigma_x}{\partial x^2} - \cancel{\frac{\partial^2\sigma_x}{\partial y^2}} - \cancel{\frac{\partial^2\sigma_x}{\partial z^2}}$$

$$\boxed{\nabla^2\theta - \frac{\partial^2\theta}{\partial x^2} - \nabla^2\sigma_x = \frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial^2\sigma_z}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial z^2}+\frac{\partial^2\sigma_z}{\partial z^2}-\frac{\partial^2\sigma_x}{\partial x^2}} \rightarrow (33)$$

from eq^n (29) & compare (33)

$$1+\nu\left[\frac{\partial^2\sigma_y}{\partial z^2}+\frac{\partial^2\sigma_z}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial^2\sigma_z}{\partial z^2}-\frac{\partial^2\sigma_x}{\partial x^2}\right] = \nabla^2\theta - \frac{\partial^2\theta}{\partial x^2} - \nabla^2\sigma_x$$

Substitute Eq^n (28) & (33) becomes

$$\boxed{(1+\nu)\left[\nabla^2\theta - \nabla^2\sigma_x - \frac{\partial^2\theta}{\partial x^2}\right] - \nu\left[\nabla^2\theta - \frac{\partial^2\theta}{\partial x^2}\right] = (1+\nu)\left[\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right]} \rightarrow (34)$$

Similarly, two analogous eq^n can be obtained from two other eq^n in (23) same as of eq^n (c)

~~that~~ that means.

~~$\frac{\partial^2 \epsilon_y}{\partial y^2}$~~ for $\frac{\partial^2\epsilon_y}{\partial z^2} + \frac{\partial^2\epsilon_x}{\partial y^2} = \frac{\partial^2\gamma_{yz}}{\partial y\,\partial z}$ the analogue eq^n obtained is

$$(1+\nu)\left[\nabla^2\theta - \nabla^2\sigma_x - \frac{\partial^2\theta}{\partial x^2}\right] - \nu\left[\nabla^2\theta - \frac{\partial^2\theta}{\partial x^2}\right] = (1+\nu)\left[\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right]$$

Same as for $\frac{\partial^2\epsilon_x}{\partial y^2} + \frac{\partial^2\epsilon_y}{\partial x^2} = \frac{\partial^2\gamma_{xy}}{\partial x\cdot\partial y}$ the analogue eq^n obtained is

$$(1+\nu)\left[\nabla^2\theta - \nabla^2\sigma_z - \frac{\partial^2\theta}{\partial z^2}\right] - \nu\left[\nabla^2\theta - \frac{\partial^2\theta}{\partial z^2}\right] = (1+\nu)\left[\frac{\partial Z}{\partial z} - \frac{\partial Y}{\partial y} - \frac{\partial X}{\partial x}\right]$$

Same as for $\frac{\partial^2\epsilon_z}{\partial x^2} + \frac{\partial^2\epsilon_x}{\partial z^2} = \frac{\partial^2\gamma_{xz}}{\partial z\cdot\partial x}$ the analogue eq^n obtained is

$$(1+\nu)\left[\nabla^2\theta - \nabla^2\sigma_y - \frac{\partial^2\theta}{\partial y^2}\right] - \nu\left[\nabla^2\theta - \frac{\partial^2\theta}{\partial y^2}\right] = (1+\nu)\left[\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial x} - \frac{\partial Z}{\partial z}\right]$$

(35)

adding these (3) equations

$$(1+\nu)\left[\nabla^2\theta - \nabla^2\sigma_x - \frac{\partial^2\theta}{\partial x^2}\right] - \nu\left[\nabla^2\theta - \frac{\partial^2\theta}{\partial x^2}\right] + (1+\nu)\left[\nabla^2\theta - \nabla^2\sigma_z - \frac{\partial^2\theta}{\partial z^2}\right] -$$

$$\nu\left[\nabla^2\theta - \frac{\partial^2\theta}{\partial z^2}\right] + (1+\nu)\left[\nabla^2\theta - \nabla^2\sigma_y - \frac{\partial^2\theta}{\partial y^2}\right] - \nu\left[\nabla^2\theta - \frac{\partial^2\theta}{\partial y^2}\right] =$$

$$(1+\nu)\left[\cancel{\frac{\partial X}{\partial x}} - \cancel{\frac{\partial Y}{\partial y}} - \cancel{\frac{\partial Z}{\partial z}} + \cancel{\frac{\partial Z}{\partial z}} - \frac{\partial Y}{\partial y} - \cancel{\frac{\partial X}{\partial x}} + \cancel{\frac{\partial Y}{\partial y}} - \frac{\partial X}{\partial x} - \frac{\partial Z}{\partial z}\right]$$

$$(1+\nu)\left[\nabla^2\theta - \nabla^2\sigma_x - \frac{\partial^2\theta}{\partial x^2} + \nabla^2\theta - \nabla^2\sigma_z - \frac{\partial^2\theta}{\partial z^2} + \nabla^2\theta - \nabla^2\sigma_y - \frac{\partial^2\theta}{\partial y^2}\right]$$

$$-\nu\left[\nabla^2\theta - \frac{\partial^2\theta}{\partial x^2} + \nabla^2\theta - \frac{\partial^2\theta}{\partial y^2} - \frac{\partial^2\theta}{\partial x^2} + \nabla^2\theta\right] = (1+\nu)\left[-\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right]$$

$$(1+\nu)\left[3\nabla^2\theta - \nabla^2(\sigma_x + \sigma_y + \sigma_z) - \theta\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right] -$$

$$\nu\left[3\nabla^2\theta - \theta\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}\right)\right] = (1+\nu)\left[-\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right]$$

$$(1+\nu)\left[3\nabla^2\theta - \nabla^2\theta - \theta\nabla^2\right] - \nu\left[3\nabla^2\theta - \theta\nabla^2\right] = (1+\nu)\left[-\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right]$$

$$\left[\nabla^2\theta + \nu\nabla^2\theta - 2\nu\nabla^2\theta\right] = -(1+\nu)\left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right]$$

$$(1-\nu)\nabla^2\theta = -(1+\nu)\left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right] \rightarrow 36$$

Substitute this Expression in eq.n 34 for $\nabla^2\theta$ then

$$(1+\nu)\left(\nabla^2\theta - \nabla^2\sigma_x - \frac{\partial^2\theta}{\partial x^2}\right) - \nu\left(\nabla^2\theta - \frac{\partial^2\theta}{\partial x^2}\right) = (1+\nu)\left(\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right) \rightarrow (34)$$

from (36)

$$\nabla^2\theta = -\frac{(1+\nu)}{(1-\nu)}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right)$$

$$(1+\nu)\left(\frac{-(1+\nu)}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) - \nabla^2\sigma_x - \frac{\partial^2\theta}{\partial x^2}\right) - \nu\left(\frac{-(1+\nu)}{(1-\nu)}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) - \frac{\partial^2\theta}{\partial x^2}\right) = (1+\nu)\left(\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right)$$

$$(1+\nu)\cancel{(1+\nu)}\left[\frac{-1}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) - \frac{\nabla\sigma_x}{(1+\nu)} - \frac{\partial^2\theta}{\partial x^2}\times\frac{1}{1+\nu}\right) - \nu\left(\frac{-1}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) - \frac{\partial^2\theta}{\partial x^2}\times\frac{1}{1-\nu}\right) = \cancel{(1+\nu)}\left(\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right).$$

$$-\frac{(1+\nu)}{(1-\nu)}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) - \nabla\sigma_x - \frac{\partial^2\theta}{\partial x^2} - \nu\left(\frac{-(1+\nu)}{1-\nu}\right)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) - $$

$$+\frac{\partial^2\theta}{\partial x^2}\times\frac{(1+\nu)}{(1-\nu)} = \frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}$$

$$-\nabla^2\sigma_x - \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial x^2}\times\frac{(1+\nu)}{(1-\nu)} = \left[\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z} + \frac{(1+\nu)}{(1-\nu)}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) + \frac{\nu(1+\nu)}{(1-\nu)}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right)\right]$$

$\downarrow$ left $\qquad\qquad$ $\downarrow$ right

Left:

$$-\nabla^2\sigma_x + \frac{\partial^2\theta}{\partial x^2}\left(-\frac{1+\nu}{1-\nu} + 1\right) \Rightarrow -\nabla^2\sigma_x + \frac{\partial^2\theta}{\partial x^2}\left(1 - \frac{1+\nu}{1-\nu}\right)$$

$$-\nabla^2\sigma_x - \frac{\partial^2\theta}{\partial x^2}\left(\frac{\cancel{1}-\nu-\cancel{1}-\nu}{1-\nu}\right) \Rightarrow \boxed{-\nabla^2\sigma_x - \frac{\partial^2\theta}{\partial x^2}\left(\frac{-2\nu}{1-\nu}\right)}$$

Right!

$$\left[\underbrace{\frac{\partial X}{\partial x}+\frac{1+\nu}{1-\nu}\frac{\partial X}{\partial x}+\frac{\nu(1+\nu)}{(1-\nu)}\frac{\partial X}{\partial x}}_{1}+\underbrace{\frac{1+\nu}{1-\nu}\frac{\partial Y}{\partial y}-\frac{\partial Y}{\partial y}+\frac{\nu(1+\nu)}{(1-\nu)}\frac{\partial Y}{\partial y}}_{2}+\underbrace{\frac{1+\nu}{1-\nu}\frac{\partial Z}{\partial z}-\frac{\partial Z}{\partial z}+\frac{\nu(1+\nu)}{(1-\nu)}\frac{\partial Z}{\partial z}}_{3}\right]$$

1
$$\frac{\partial X}{\partial x}\left[1+\frac{1+\nu}{1-\nu}+\frac{\nu(1+\nu)}{1-\nu}\right] \Rightarrow \frac{(1-\nu)+(1+\nu)+\nu+\nu^2}{(1-\nu)}\frac{\partial X}{\partial x}$$

$$\Rightarrow \frac{1-\cancel{\nu}+1+\cancel{\nu}+\nu+\nu^2}{(1-\nu)}\frac{\partial X}{\partial x} \Rightarrow \boxed{\frac{2+\nu+\nu^2}{(1-\nu)}\frac{\partial X}{\partial x}}$$

2
$$\frac{\partial Y}{\partial y}\left[\frac{1+\nu}{1-\nu}-1+\frac{\nu+\nu^2}{1-\nu}\right] \Rightarrow \frac{\cancel{1}+\nu-\cancel{1}+\nu+\nu+\nu^2}{(1-\nu)}\frac{\partial Y}{\partial y}$$

$$\Rightarrow \boxed{\frac{3\nu+\nu^2}{1-\nu}\frac{\partial Y}{\partial y}}$$

3
$$\frac{\partial Z}{\partial z}\left[\frac{1+\nu}{1-\nu}-1+\frac{\nu+\nu^2}{1-\nu}\right] \Rightarrow \frac{3\nu+\nu^2}{1-\nu}\frac{\partial Z}{\partial z}$$

1 + 2 + 3

$$\frac{1}{1-\nu}\left[2+\nu+\nu^2\frac{\partial X}{\partial x}+\frac{3\nu+\nu^2}{}\frac{\partial Y}{\partial y}+\frac{3\nu+\nu^2}{}\frac{\partial Z}{\partial z}\right]$$

$$\frac{\nu}{1-\nu}\left[\left(\frac{2}{\nu}+1+\nu\right)\frac{\partial X}{\partial x}+(3+\nu)\frac{\partial Y}{\partial y}+(3+\nu)\frac{\partial Z}{\partial z}\right]$$

$$\frac{\nu}{1-\nu}\left[\frac{2}{\nu}\frac{\partial X}{\partial x}+\frac{\partial X}{\partial x}+\nu\frac{\partial X}{\partial x}+3\frac{\partial Y}{\partial y}+\nu\frac{\partial Y}{\partial y}+3\frac{\partial Z}{\partial z}+\nu\frac{\partial Z}{\partial z}\right]$$

from this Eqn we can obtain means

Left = Right we to get

$$\boxed{\nabla^2\sigma_x + \frac{1}{1+\nu}\frac{\partial^2\theta}{\partial x^2} = \frac{-\nu}{1-\nu}\left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right] - 2\frac{\partial X}{\partial x}} \rightarrow (37)$$

We can obtained 3 eqn of this kind these 3 remaining converted into

$$\boxed{\nabla^2\tau_{yz} + \frac{1}{1+\nu}\frac{\partial^2\theta}{\partial y\cdot\partial z} = -\left(\frac{\partial Z}{\partial y} + \frac{\partial Y}{\partial z}\right)} \rightarrow (38)$$

if there is no body force ~~it~~ becomes (37) & (38)

$$\left.\begin{aligned}(1+\nu)\nabla^2\sigma_x + \frac{\partial^2\theta}{\partial x^2} &= 0\\ (1+\nu)\nabla^2\sigma_y + \frac{\partial^2\theta}{\partial y^2} &= 0\\ (1+\nu)\nabla^2\sigma_z + \frac{\partial^2\theta}{\partial z^2} &= 0\end{aligned}\right\} \rightarrow (39)$$

$$\left.\begin{aligned}(1+\nu)\nabla^2\tau_{yz} + \frac{\partial^2\theta}{\partial y\cdot\partial z} &= 0\\ (1+\nu)\nabla^2\tau_{xz} + \frac{\partial^2\theta}{\partial x\cdot\partial z} &= 0\\ (1+\nu)\nabla^2\tau_{xy} + \frac{\partial^2\theta}{\partial x\cdot\partial y} &= 0\end{aligned}\right\} \rightarrow (40)$$

~~these~~ are the six conditions of compatibility to find the stress components.

~~* Determination of Displacements:-~~

~~* Equation of Equilib~~

6

* Equation of Equilibrium interms of displacements:-

1. In this unit Eq. (18) & (1),(2),(3) to eliminate the stress components by using hooke's law

2. In general $$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0 \quad \boxed{\text{from } 18}$$

In the first unit (7)th eq.

$$\boxed{\sigma_x = \lambda e + 2G\,\epsilon_x} \qquad \epsilon_x = \frac{\partial u}{\partial x}$$

In the first unit 3(a)

$$\tau_{xy} = G\cdot\gamma_{xy} \;; \quad \tau_{xz} = G\cdot\gamma_{xz}$$

$$\boxed{\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}} ; \quad \boxed{\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}}$$

Substitute these (3) equations in 18th eq.

here $\sigma_x = \lambda e + 2G\frac{\partial u}{\partial x}$; $\tau_{xy} = G\cdot\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$; $\tau_{xz} = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$

$$\frac{\partial}{\partial x}\left(\lambda e + 2G\cdot\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) + \frac{\partial}{\partial z}\left(G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right) + X = 0$$

$$\lambda\frac{\partial e}{\partial x} + 2G\frac{\partial^2 u}{\partial x^2} + G\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x\cdot\partial y}\right) + G\left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial z\cdot\partial x}\right) + X = 0$$

$$\lambda\frac{\partial e}{\partial x} + G\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\cdot\partial y} + \frac{\partial^2 w}{\partial z\cdot\partial x}\right) + X = 0$$

$$\lambda\frac{\partial e}{\partial x} + G\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\cdot\partial y} + \frac{\partial^2 w}{\partial x\cdot\partial z}\right) + \underbrace{G\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}_{G\cdot\nabla^2 u} + X = 0$$

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actually $e = \varepsilon_x + \varepsilon_y + \varepsilon_z$

$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\boxed{\frac{\partial e}{\partial x} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 w}{\partial x \cdot \partial z}}$$ so Substitute this eqn

$$\lambda \frac{\partial e}{\partial x} + G \frac{\partial e}{\partial x} + G \times \nabla^2 u + X = 0$$

$$\boxed{(\lambda + G) \frac{\partial e}{\partial x} + G \nabla^2 u + X = 0} \rightarrow (41)$$

Same as

$$\boxed{(\lambda + G) \frac{\partial e}{\partial x} + G \cdot \nabla^2 v + Y = 0} \rightarrow (42)$$

$$\boxed{(\lambda + G) \frac{\partial e}{\partial x} + G \nabla^2 w + Z = 0} \rightarrow (43)$$

if no body forces

$$\boxed{(\lambda + G) \frac{\partial e}{\partial x} + G \nabla^2 u = 0 ; \quad (\lambda + G) \frac{\partial e}{\partial x} + G \cdot \nabla^2 v = 0 \quad (\lambda + G) \frac{\partial e}{\partial x} + G \cdot \nabla^2 w = 0} \rightarrow (44)$$

Differentiating these eqns first w.r.t x, second w.r.t y, third w.r.t z

~~Differentiating~~

adding them together we find $\rightarrow \boxed{(\lambda + 2G) \nabla^2 e = 0}$ which satisfies this eqn

$$\boxed{\frac{\partial^2 e}{\partial x^2} + \frac{\partial^2 e}{\partial y^2} + \frac{\partial^2 e}{\partial z^2} = 0}$$

* Principle of Superposition:-

① The Solution of a problem of a given elastic solid with given Surface & body forces requires us to det; stress components, (or) displacements, that satisfy the differential Equations & the boundary conditions.

② If we choose to work with stress components, we have to satisfy the Equations of equilibrium, compatibility conditions & the boundary conditions.

③ Let $\sigma_x, \ldots, \tau_{xy} \ldots$ be the stress components due to surface forces $\bar{X}, \bar{Y}, \bar{Z}$ & body forces X, Y, Z

④ Let $\sigma_x' \ldots, \tau_{xy}' \ldots$ be the stress components to the same elastic solid due to surface forces $\bar{X}', \bar{Y}', \bar{Z}'$ & body forces X', Y', Z'. Then the stress components $\sigma_x + \sigma_x', \ldots, \tau_{xy} + \tau_{xy}' \ldots$ will represent the stress due to surface forces $\bar{X} + \bar{X}' \ldots$ & the body forces $X + X' \ldots$

⑤ This holds because all the differential equations & boundary conditions are linear.

then
$$\frac{\partial \sigma_x'}{\partial x} + \frac{\partial \tau_{xy}'}{\partial y} + \frac{\partial \tau_{xz}'}{\partial z} + X = 0$$

we find
$$\frac{\partial}{\partial x}\left(\sigma_x + \sigma_x'\right) + \frac{\partial}{\partial y}\left(\tau_{xy} + \tau_{xy}'\right) + \frac{\partial}{\partial z}\left(\tau_{xz} + \tau_{xz}'\right) + X + X' = 0$$

and Similarly

$$\bar{X} + \bar{X}' = (\sigma_x + \sigma_x')l + (\tau_{xy} + \tau_{xy}')m + (\tau_{xz} + \tau_{xz}')n$$

⑥ The compatibility conditions can be ~~of~~ combined to the same manner. The complete set of equations shows that $\sigma_x + \sigma_x'$, ... $\tau_{xy} + \tau_{xy}'$ .... satisfy all the eqns & conditions determining the stress due to forces $\bar{X} + \bar{X}'$, ..., $X + X'$.... This is an instance of the "principle of superposition".

⑦ In deriving our equations of equilibrium & boundary conditions, we made no distinction b/w the position & form of the element before loading. & its position & form after loading

⑧ As a consequence, our equations & the conclusions drawn from them are valid only so long as the ~~[illegible]~~ small displacements in the deformation do not effect substantially the action of the external forces.

⑨ There are cases, ~~where~~ however, in which the deformation must be take into account.

⑩ Then the justification of the "principle of superposition" given above fails.

<u>For Ex:-</u>

The beam under simultaneous thrust & lateral load affords an example of this kind & many other case in considering the elastic stability of thin walled structure.

* Uniqueness of Solution:-

① We consider now whether our Equations can have more than one solution corresponding to given surface & body forces.

② Let $\sigma_x' \ldots, \tau_{xy}' \ldots$ represent a solution for loads $\bar{X} \ldots, X$ & let $\sigma_x'' \ldots, \tau_{xy}'' \ldots$ represent a second solution for same loads $\bar{X} \ldots, X \ldots$

③ Then for 1st solution we have

$$\frac{\partial \sigma_x'}{\partial x} + \frac{\partial \tau_{xy}'}{\partial y} + \frac{\partial \tau_{xz}'}{\partial z} + X = 0$$

- - - - - - - - -

$$\bar{X} = \sigma_x' l + \tau_{xy}' m + \tau_{xz}' n$$

- - - - - - - - -

④ & also the conditions of compatability for the second solution $\frac{\partial \sigma_x''}{\partial x} + \frac{\partial \tau_{xy}''}{\partial y} + \frac{\partial \tau_{xz}''}{\partial z} + X = 0$

- - - - - - - - -

$$\bar{X} = \sigma_x'' l + \tau_{xy}'' m + \tau_{xz}'' n$$

- - - - - - - - -

By subtraction we find

⑤ $$\frac{\partial (\sigma_x' - \sigma_x'')}{\partial x} + \frac{\partial (\tau_{xy}' - \tau_{xy}'')}{\partial y} + \frac{\partial (\tau_{xz}' - \tau_{xz}'')}{\partial z} = 0$$

- - - - - - - - -

$$0 = (\sigma_x' - \sigma_x'')l + (\tau_{xy}' - \tau_{xy}'')m + (\tau_{xz}' - \tau_{xz}'')n$$

- - - - - - - - -

in which all External forces vanish.

(6) The conditions of compatibility will also be satisfied by the corresponding strain components $\epsilon_x' - \epsilon_x'' \ldots\ldots,$ $\gamma_{xy}' - \gamma_{xy}''$.

(7) Thus this stress distribution is one that corresponds to zero surface & body forces

(8) The work done by these forces during loading is zero,
~~& it follows that~~ $\cancel{\iiint V_0 \, dx \, dy \, dz}$ ~~vanishes.~~
This requires that each of the strain components $\epsilon_x' - \epsilon_x'', \ldots, \gamma_{xy}' - \gamma_{xy}''; \ldots$ should be zero.

(9) & consequently the two states of stress $\sigma_x' \ldots \tau_{xy}' \ldots$ and $\sigma_x'' \ldots, \tau_{xy}'' \ldots$ are therefore identical.

(10) That is, the equations can yield only one solution corresponding to given loads.

(11) The proofs of uniqueness of solution was based on the assumption that the strain energy, & hence stresses, in a body disappear when it is free of external forces.

(12) however there are case when "initial stresses" may exist in a body when external forces are absent.

(13) Ex:- this kind was encountered in studying the circular ring. if a portion of the ring b/w two adjacent c/s is cut out, & the ends of the ring are joined again by welding, a ring with initial stresses is obtained.

* Reciprocal theorem :-

(16)

① We ~~know~~ consider a given elastic body under one set of given surface forces $\bar{X}, \bar{Y}, \bar{Z}$. & body forces X', Y', Z' & ~~&~~ regard displacements, strains & stresses as known.

② These will be denoted by $u', \epsilon_x', \gamma_{xy}', \sigma_x', \tau_{xy}'$ etc., the independently, we consider a second set of forces $\bar{X}''$ etc., ~~&~~ & indicating results for this second problem $u'', \epsilon_x'', \gamma_{xy}'', \sigma_x'', \tau_{xy}''$

③ We have then two distinct solutions of two distinct problems. But the fact that they refer to the same elastic body is a relation b/w them.

here we establish, one aspect of this relation - the Reciprocal theorem:

④ From the two solutions we can form, purely as mathematical operation, the quantity "T" defined by

$$\boxed{'T'' = \int(\bar{X}'u'' + \bar{Y}'v'' + \bar{Z}'w'')\,dS + \int(X'u'' + Y'v'' + Z'w'')\,d\tau} \quad (45)$$

Interchanging single and double primes throughout we can also form

$$\boxed{''T' = \int(\bar{X}''u' + \dots + \dots)\,dS + \int(X''u' + \dots + \dots)\,d\tau}$$

$$\boxed{'T'' = ''T'} \quad (46)$$

~~For proof we require again the divergence theorem~~

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~~Consider~~ $\cancel{\int \bar{X} u'' dS}$

~~i.e.,~~ $\cancel{\int \left(l \sigma_x' + m \tau_{xy}' + n \tau_{xz}' \right) u'' dS}$

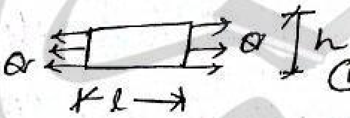
~~Consider~~ $\cancel{\boxed{U'' \sigma_x' = U}}$ $\cancel{\boxed{u'' \tau_{xy}' = V}}$ $\cancel{\boxed{u'' \tau_{xz}' = W}}$

~~actually~~ $\cancel{\frac{\partial \sigma_x'}{\partial x} + \frac{\partial \tau_{xy}'}{\partial y} + \frac{\partial \tau_{xz}'}{\partial z} + X' = 0}$

(5) The theorem can be immediately extended to the dynamical case by including the inertia forces as body forces.

(6) we have many important applications.

__Ex1:__ Consider first a uniform bar compressed by two equal & opposite forces i.e.,

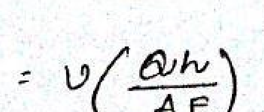


(1)

(7) the problem of finding the stresses produced by these forces is a complicated one.

(8) but suppose we are intrested not in the stresses but in the total elongation "δ" of the bar.

for this condition we consider the stress condition as



(2) for the first case lateral contraction

$\delta_1 = \nu \left(\frac{Qh}{AE} \right)$ (P)

actually from given reciprocal theorem $P \nu \left(\frac{Qh}{AE} \right) = Q \cdot \delta$.

& the elongation $\delta = P \frac{\nu h}{AE}$ i.e., $\delta = \nu \frac{Ph}{AE}$

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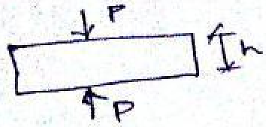
Ex 2:

→ Let us calculate the reduction Δ in volume of an elastic body produced by two equal & opposite forces P



→ As a second state we take the same body submitted to the action of uniformly distributed pressure "p".

→ In this latter case we will have at each point of the body a uniform compression in all directions of the magnitude $\frac{(1-2\nu)p}{E}$ and the distance "l" b/w the points of application A & B will be diminished by the amount $\frac{(1-2\nu)pl}{E}$.

→ The reciprocal theorem applied to the two states

~~$P\frac{(1-2\nu)p}{E} = \Delta p$~~ and $P\frac{(1-2\nu)pl}{E} = \Delta p$

Δ the reduction in the volume of the body

therefore $\Delta = \frac{Pl(1-2\nu)}{E}$

UNIT-V

Torsion of Prismatical bars - Bars with elliptical c/s - Other elementary solution - Membrane analogy - Torsion of rectangular bars - Solution of torsional problems by energy method.

* Torsion of Straight bars:-

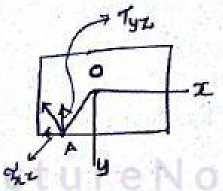
1) We know the Exact solution of the torsional problem for a circular shaft obtained if we assume that the c/s of the bar remain plane & rotate without any distortion during twist.

2) This theory developed by Coloumb was applied later by Navier to bars of non-circular c/s.

3) Making the above assumption he arrived at the erroneous conclusions that, for a given torque, the angle of twist of bars is inversely proportional to the centroidal polar moment of inertia of the c/s..

4) and also max: shearing stress occurs at the points most remote from the centroid of the c/s.

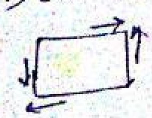
5) Take for Instance,
a bar of rectangular c/s



a From Navier's assumptions it follows that at any point A on the boundary the shearing stress should act in the direction perpendicular to the radius 'OA'

6) Resolving this stress into two components τ_{xz}, τ_{yz}

7) On the element of lateral surface of the bar at the point 'A'



which is in contradiction with the assumption that the lateral surface of the bar is free from External forces, the twist being produced by couples

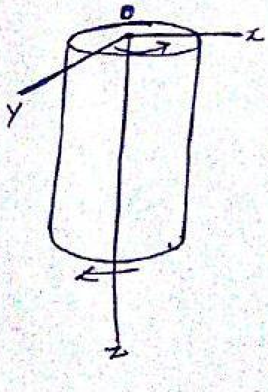
(8) A simple Experiment with a rectangular bar, shows that the c/s of the bar do not remain plane during torsion & that the distortions of rectangular elements on the surface of the bar are greatest at the middles of the sides i.e, at the points which are nearest to the axis of the bar.

(9) The correct solution of the problem of torsion of bars by couples applied at the ends was given by Saint - Venant.

(10) He used the so - called "Semi - Inverse method".
he made certain assumptions as to the deformation of the twisted bar & showed that with these assumptions he could satisfy the equations equilibrium & boundary conditions.

(11) Consider a uniform bar of any c/s twisted by couples applied at the ends

(12) Guided by the solution for a circular shaft Saint - Venant assumes that the deformation of the twisted shaft consists of 1) rotation of c/s, of the shaft (for circular shaft)
2) Warping of the c/s. which is the same (for all c/s).



(13) Taking the origin of co-ordinates in an end c/s in the fig.

(14) the displacements corresponding to rotation of c/s are

$u = -\theta_z \, y \rightarrow (a) \quad v = \theta_z \, x \rightarrow (b)$

θ_z is the angle of rotation of c/s at a dist. z from the origin.

(15) the warping of c/s is defined by a function ψ by writing

$$\omega = \theta \psi (x, y) \rightarrow (c)$$

We calculate the components of strain

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xy} = 0$$

(16) $\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \theta\left(\frac{\partial \psi}{\partial x} - y\right)$

$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \theta\left(\frac{\partial \psi}{\partial y} + x\right)$ $\rightarrow$ (1)

the corresponding components of stress from 1st unit eqn (3)

$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$

τ_{xz} & τ_{yz} from eqn 2(a) from 1st unit

~~$\gamma_{xy} = \frac{\tau_{xz}}{}$~~ $\boxed{\gamma_{xz} = \frac{1}{G}\tau_{xz} \quad ; \quad \gamma_{yz} = \frac{1}{G}\tau_{yz}}$

$\tau_{xz} = G \times \gamma_{xz}$
$= G \times \theta\left(\frac{\partial \psi}{\partial x} - y\right)$

$\tau_{yz} = G \times \gamma_{yz}$
$= G \times \theta\left(\frac{\partial \psi}{\partial y} + x\right)$ $\rightarrow$ (2)

(17) It can be seen that with the assumptions regarding the deformation, there will be no normal stresses acting b/w the longitudinal fibers of the shaft.

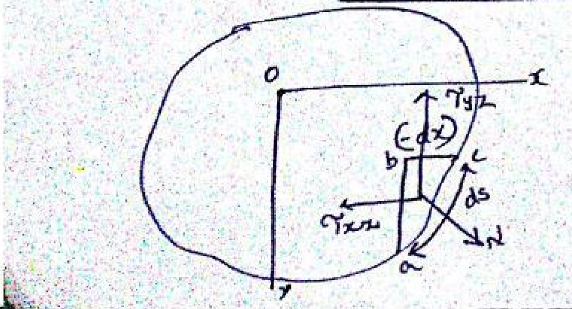
(18) There also will be no distortion in the planes of c/s, since $\epsilon_x = \epsilon_y = \gamma_{xy}$ are vanish. Only we have τ_{xz} & τ_{yz}

Substituting expressions (2) in these eqn & neglecting body forces we find that the function ψ must satisfy eqn

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0} \rightarrow (2a)$$

(19) consider the boundary conditions (1),(2),(3) in unit (4) for the lateral surface of the bar, which is free from external forces & has normals perpendicular to the z-axis, we have $\bar{X} = \bar{Y} = \bar{Z} = 0$ & $n = 0$ so ....

$\boxed{\tau_{xz}\, l + \tau_{yz}\, m = 0} \rightarrow 3 \rightarrow \boxed{\tau_{xz}\, l + \tau_{yz}\, m = 0}$



$l = \frac{dy}{ds} = \cos Nx$

$m = \cos Ny = -\frac{dx}{ds}$

(20) Eq^n (3) becomes

$$\left(\frac{\partial \varphi}{\partial x} - y\right)\frac{dy}{ds} - \left(\frac{\partial \varphi}{\partial y} + x\right)\frac{dx}{ds} = 0$$

if we seen in (2) eq^n

τ_{xz} & τ_{yz}

(21) $\tau_{xz} = G\theta\left(\frac{\partial \varphi}{\partial x} - y\right)$ we may take this as zero if we diff^z w.r.t. z

$\frac{\partial \tau_{xz}}{\partial z} = 0$ means τ_{xz} independent on "z"

same as $\frac{\partial \tau_{yz}}{\partial z} = 0$ same $\boxed{\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0}$

because if seen eq^n (2a) satisfy this equation

(22) We can express the $\tau_{xz} = \frac{\partial \phi}{\partial y}$

$\tau_{yz} = \frac{-\partial \phi}{\partial x}$

ϕ is a stress function, it is a function of x & y

$$\frac{\partial \phi}{\partial y} = \tau_{xz} = G\theta\left(\frac{\partial \varphi}{\partial x} - y\right)$$

$$\frac{-\partial \phi}{\partial x} = \tau_{yz} = G\theta\left(\frac{\partial \varphi}{\partial y} + x\right)$$

for finding the stress function,

(23) $\boxed{\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2} = -2G\theta}$ (4) this eq^n should satisfy if u want to find the stress function.

the boundary condition (3) becomes

$$\boxed{\frac{\partial \phi}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial \phi}{\partial x} \cdot \frac{dx}{ds} = \frac{d\phi}{ds} = 0} \longrightarrow 4(a)$$

This shows that the stress function ϕ must be constant along the B.C of c/s

(24) thus the determination of the stress distribution over a c/s of a twisted bar consists in finding the function ϕ should satisfy the eq^n (4)

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(25) let us consider the conditions of twisted bars.
hence $l = m = 0$; $n = \pm 1$ eqn 1, 2, 3 In unit (4) becomes

$$\boxed{\bar{X} = \pm \tau_{xz} \qquad \bar{Y} = \pm \tau_{yz}} \rightarrow (5)$$

(26) In which + sign ~~indicates~~ should be taken for the end of the bar for which the external normal has the direction of positive z axis,

(27) We see that over the ends a the shearing forces are distributed in the same manner as the shearing stresses over the c/s of the bar.

(28) It is easy to prove that these forces give us a torque

We know $\tau_{xz} = \partial\phi/\partial y$

$\tau_{yz} = -\partial\phi/\partial x$

from eqn (5)

observing ~~at the~~ that ϕ at the boundary is zero

then

$$\iint \bar{X}\, dx \cdot dy = \iint \tau_{xz}\, dx \cdot dy = \iint \frac{\partial\phi}{\partial y}\, dx \cdot dy$$

$$= \int dx \int \frac{\partial\phi}{\partial y}\, dy = 0$$

Same as

$$\iint \bar{Y}\, dx\, dy = \iint \tau_{yz}\, dx \cdot dy = -\iint \frac{\partial\phi}{\partial x}\, dx \cdot dy$$

$$= -\int dy \int \frac{\partial\phi}{\partial x} \cdot dx = 0$$

Thus the resultant of the forces distributed over the ends of the bar is zero & these forces represent a couple the magnitude of

$$\boxed{M_t = \iint (\bar{Y}x - \bar{X}y)\, dx \cdot dy = -\iint \frac{\partial\phi}{\partial x} x\, dx \cdot dy - \iint \frac{\partial\phi}{\partial y} \cdot y \cdot dx\, dy} \rightarrow (6)$$

(29) Integrating this by parts
&
observing that $\phi = 0$ at the boundary

$$\boxed{M_t = 2\iint \phi\, dx\, dy} \rightarrow 6(a)$$

each of the integrals in the last member in eq (6)
i.e. $\left(-\iint \frac{\partial \phi}{\partial x}\, dx\, dy\right)$
contributing one half of this torque

(30) Thus half the torque is due to τ_{xz} & ~~half other~~ half to τ_{yz}

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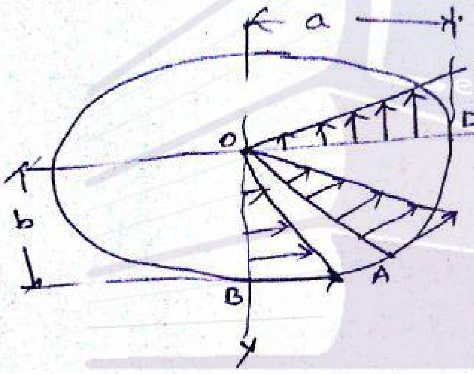
* Bars with elliptic cross-section.



1) Let the boundary of c/s be given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

2) we know $\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta = F}$

& from eq (4a) boundary condition are satisfied by taking the stress function in the form

$$\phi = m\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

"m" is a constant

substituting ϕ value in $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = F$

$$\frac{\partial^2}{\partial x^2}\left[m\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\right] + \frac{\partial^2}{\partial y^2}\left[m\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\right] = F$$

$$\cancel{m\left[\frac{2x}{a^2} + \frac{2y}{b^2}\right] = F}$$

$$\cancel{m = F \times a^2 b^2}$$

$$m\left[\frac{2}{a^2} + \frac{2}{b^2}\right] = F$$

$$\boxed{\frac{F \times a^2 b^2}{2(a^2+b^2)} = m} \rightarrow (7)$$

hence

$$\phi = m\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

$$= F\,\frac{a^2 b^2}{2(a^2+b^2)}\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right]$$

The Magnitude of const. F will now be determined

from $M_t = 2\iint \phi \, dx \cdot dy$

$$M_t = \cancel{2}\iint F\,\frac{a^2 b^2}{\cancel{2}(a^2+b^2)}\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right] dx \cdot dy$$

$$M_t = \frac{a^2 b^2 F}{a^2+b^2}\iint\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) dx \cdot dy$$

$$= \frac{a^2 b^2 F}{a^2+b^2}\left[\iint \frac{1}{a^2}(x^2)\,dx \cdot dy + \iint \frac{1}{b^2}(y^2)\,dx \cdot dy - \iint dx \cdot dy\right]$$

Since

$$\iint x^2 \cdot dx \cdot dy = \int \frac{x^3}{3} dy$$

$$= \frac{x^3}{3} \times y \quad ? = \boxed{\frac{\pi}{4} a^3 b = I_y}$$

$$\iint y^2\, dx \cdot dy = \int \frac{y^3}{3} dx = \frac{y^3 x}{3} \;?\; \frac{\pi}{4} b^3 a$$

$$\iint dx \cdot dy = \int x \cdot dy = xy \;?\; \pi ab$$

Substituting all values

$$M_t = \frac{a^2 b^2 F}{a^2+b^2}\cancel{\left[\frac{1}{a^2} \cdot \frac{x^3 y}{3} + \frac{1}{b^2}\frac{y^3 x}{3} - xy\right]}\left(\frac{\pi}{4} ba^3 + \frac{\pi}{4} ab^3 - \pi ab\right)$$

$$M_t = -\frac{\pi a^3 b^3 F}{2(a^2+b^2)}$$

$$\boxed{F = \frac{-2 M_t (a^2+b^2)}{\pi a^3 b^3}} \rightarrow 6a$$

→ substitute in ϕ

$$\phi = -\frac{M_t}{\pi ab}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right) \longrightarrow (7)$$

Substitute (7) in $\tau_{xz} = \frac{\partial\phi}{\partial y}$

$$\tau_{yz} = \frac{-\partial\phi}{\partial x}$$

then $$\boxed{\tau_{xz} = \frac{-2M_t y}{\pi ab^3} \quad ; \quad \tau_{yz} = \frac{2M_t x}{\pi a^3 b}} \rightarrow (8)$$

⇒ the ratio of the stress components is $\propto$ y & x

⇒ Along the vertical axis OB the stress component τ_{yz} is zero the resultant stress is τ_{xz}

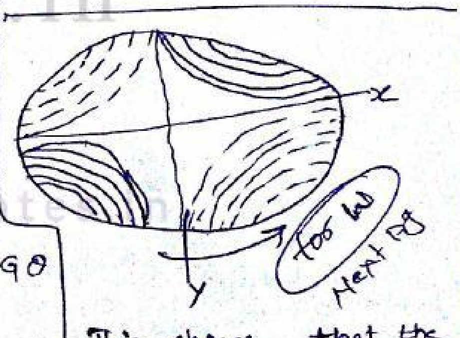
⇒ same as through OD the resultant stress is τ_{yz}

it is evident that the max: stress is at the boundary & substituting y = b in eqn (8)

$$\tau_{xz} = \tau_{max} = \frac{-2M_t y}{\pi ab^3} = \frac{-2M_t b}{\pi ab^3}$$

$$= \frac{-2M_t}{\pi ab^2}$$

⇒ Suppose if a=b this formula coincides with well-known formula for a circular c/s.



This shows that the contours lines for the warped c/s

⇒ actually F = −2Gθ

substituting ~~F value~~ (6a) value in F = −2Gθ

$$\frac{\cancel{-}\cancel{2}M_t(a^2+b^2)}{\pi a^3 b^3} = \cancel{-}\cancel{2}G\theta$$

$$\boxed{\theta = \frac{M_t(a^2+b^2)}{\pi a^3 b^3 G}} \longrightarrow (9)$$

~~from eq^n (4)~~

$$\theta = M_t \frac{a^2+b^2}{\pi a^3 b^3 G}$$

$$\frac{\pi a^3 b^3 G}{a^2+b^2}\theta = M_t$$

$$\frac{\pi a^3 b^3 G}{a^2+b^2} = M_t \times \frac{1}{\theta}$$

We know

$$\frac{C\theta}{l} = \frac{\tau}{R}$$

$$C\theta = \frac{\tau}{R} \times l$$

$$C = \frac{\tau}{R} \times \frac{l}{\theta}$$

$$= \frac{\frac{\tau l}{R} \rightarrow M}{\theta}$$

so $\dfrac{\pi a^3 b^3 G}{a^2+b^2} = C$

~~Take $A = \pi ab$~~

~~$I_P = \frac{\pi a b^3}{4} + \frac{\pi b a^3}{4}$~~

$$\boxed{C = \frac{\pi a^3 b^3 G}{a^2+b^2} \quad (\text{or}) \quad \frac{G A^4}{4\pi^2 \times I_P}}$$

$A = \pi ab \qquad I_P = \dfrac{\pi a b^3}{4} + \dfrac{\pi b a^3}{4}$

The components u & v are given in (a) & (b) of this unit

~~but the displacement w is found from~~

~~$\tau_{xz} = \frac{2M_t y}{\pi a b^3}$~~ ~~$\tau_{yz} = \frac{2M_t x}{\pi a^3 b}$~~

~~$\theta = M_t \frac{a^2+b^2}{\pi a^3 b^3 G}$~~

* Other elementary solutions:-

① In studying the torsional problem, Saint-Venant discussed several solutions of eqn $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=F$.

② To solve the problem:

Let us represent the stress function in the form

$$\phi = \phi_1 + \frac{F}{4}(x^2+y^2) \longrightarrow (10)$$

from $\boxed{\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=F}$

$$\frac{\partial^2\phi_1}{\partial x^2}+\frac{\partial^2\phi_1}{\partial y^2}=0$$

along the boundary condition from eqn (4)

the stress function ϕ is constant along the boundary condition

$$\phi_1 + \frac{F}{4}(x^2+y^2) = \text{const.}$$

not completed

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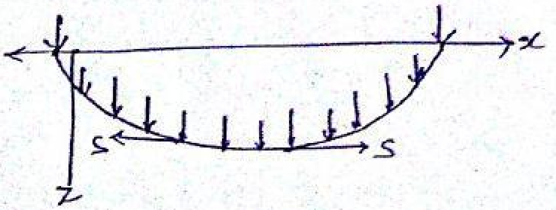
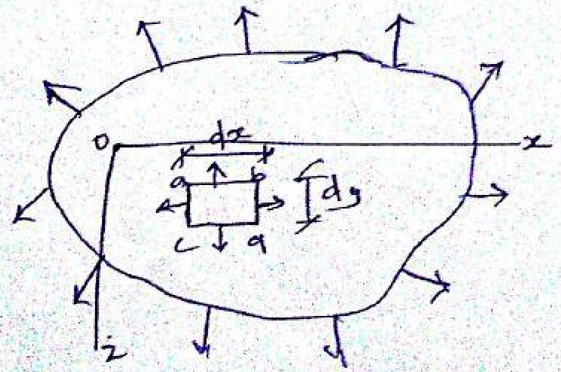
* Membrane Analogy:- (for torsional problems)

① Imagine a homogeneous membrane supported at the edges, that of the c/s of the twisted bar, subjected to uniform tension at the edges & uniform lateral pressure.





② if "q" is the pressure/unit area of the membrane & "S" is the uniform tension per unit length

③ the tensile forces acting on the sides ad & bc, In case of small deflections of the membrane, a resultant in the upward direction

$$-S\left(\frac{\partial^2 z}{\partial x^2}\right)dx\cdot dy$$

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(4) In the same manner, the tensile forces acting on the other two sides of the element give the resultant

$$-S\left(\frac{\partial^2 z}{\partial y^2}\right) dx \cdot dy.$$

(5) the eqn of Equilibrium of element is

$$q\, dx \cdot dy + S\frac{\partial^2 z}{\partial x^2} dx \cdot dy + S \cdot \frac{\partial^2 z}{\partial y^2} dx \cdot dy = 0$$

then

$$\cancel{dx \cdot dy}\, S\left[\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right] = -q\,\cancel{dx \cdot dy}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{-q}{S}$$

(6) At the boundary the deflection of the membrane is zero.

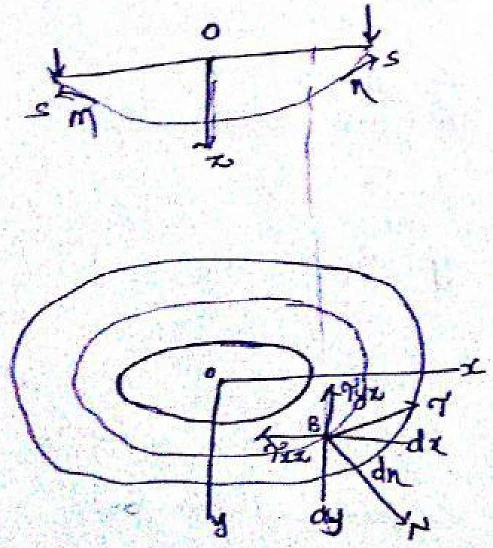
Comparing eqn $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{-q}{S}$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = F$$

& stress function @ boundary of c/s is constant.

(7) by replacing $-\frac{q}{S}$ as F

$$F = -2G\theta$$



(8) Consider any point B on the membrane, the defn of the membrane along the contour line through this point is const. &

$$\frac{\partial z}{\partial s} = 0$$

(9) The corresponding eqn for the stress function ϕ is

$$\frac{\partial \phi}{\partial s} = 0 \quad \text{i.e.,} \quad \boxed{\tau_{xz}\frac{dy}{ds} - \tau_{yz}\cdot\frac{dx}{ds} = 0}$$

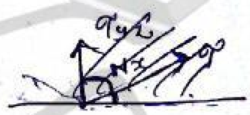
(10) $\tau_{xz}\dfrac{dy}{ds} - \tau_{yz}\dfrac{dx}{ds} = 0$

This express that the projection of the resultant shearing stress at point "B" on the Normal N to the contour line is zero

(11) we may conclude that the shearing stress at a point "B" in the twisted bar in the direction of the tangent to the contour line through this point. i.e,

(12) The curves drawn in the c/s of the twisted bar, in such a manner that the resultant shearing stress at any point of the curve in the direction of the tangent to the curve, are called "Lines of shearing stress".

(13) The Magnitude of the Resultant stress τ at B is obtained by projecting on the tangent the stress components τ_{xz} & τ_{yz} then



$$\tau = \tau_{yz}\cos(Nx) - \tau_{xz}\cos(Ny)$$

$\cos(Nx) = \dfrac{dx}{dn}$

$\cos(Ny) = \dfrac{dy}{dn}$

Substituting

$$\tau_{xz} = \frac{\partial\phi}{\partial y} \qquad \tau_{yz} = \frac{-\partial\phi}{\partial x}$$

then $\tau = \cancel{\dfrac{\partial\phi}{\partial y}\times\dfrac{dx}{dn} + \dfrac{\partial\phi}{\partial x}\dfrac{dy}{dn}}$

$$\tau = -\frac{\partial\phi}{\partial x}\cdot\frac{dx}{dn} + \frac{\partial\phi}{\partial y}\cdot\frac{dy}{dn}$$

$$\tau = \frac{-d\phi}{dn}$$

Thus, the Magnitude of the shearing stress at B is given by the Max. slope of Membrane at this point.

(14) Let us consider now the equilibrium condition of the portion mn of the membrane bounded by contour line.

(15) The slope of membrane along this line α τ

i.e., ~~$\tau\left(\frac{q}{S}\right)$ $2q\theta$~~ ~~$q \alpha F$~~ ~~actually~~ ~~$F = 2G\theta$~~ ~~$\frac{Fq}{S}$ $2G\theta$~~

that is equal to $\tau\left(\frac{q}{S}\right)\left(\frac{1}{2G\theta}\right)$

the eqn of equilibrium $\int S\left(\tau\frac{q}{S}\frac{1}{2G\theta}\right)ds = q \cdot A$

(or) $\boxed{\int \tau\, ds = 2G\theta \cdot A}$

from this the avg. value of shearing stress along contour line can be obtained.

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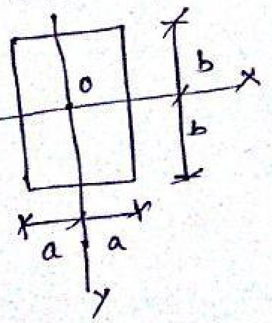
* **Torsion of Rectangular Bars:**

(1) Using the membrane analogy, the problem reduce to finding the deflections of uniformly loaded rectangular membrane



2) These deflections must satisfy the $\rightarrow$ (a)

$$\boxed{\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2} = -\frac{q}{S}}$$

& be zero at the boundary

3) The condition of symmetry w.r.t. y axis the B.C at the sides $x = \pm a$ of the rectangle are satisfied by taking "z" in the form of a series.

$$\boxed{z = \sum_{n=1,3,5,\ldots}^{\infty} b_n \cos\frac{n\pi x}{2a} Y_n} \rightarrow (b)$$

In which $b_1, b_3 \ldots$ are constant co-efficients & $Y_1, Y_3 \ldots$ are functions of y only.

~~4) Substitute (b) in eqn (a) & observing that the constant on the right side of (a) can be represented for $-a < x < a$ by the fourier series~~

$$\cancel{\frac{-q}{s} = -\sum_{n=1,3,5}^{\infty} \frac{q}{s}\frac{4}{n\pi}(-1)^{(n-1)/2}\cos\frac{n\pi x}{2a}} \rightarrow (c)$$

~~5) We arrive at the following eqn for determining Y_n~~

$$\cancel{Y_n'' - \frac{n^2\pi^2}{4a^2}Y_n = \frac{-q}{s}\frac{4}{n\pi b_n}(-1)^{(n-1)/2}}$$

4) from (a) & (b)

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{-q}{S} \qquad z = \sum_{n=1,3,5\ldots}^{\infty} b_n \cos\frac{n\pi x}{2a} Y_n$$

Y_n = functions of

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\sum_{n=1,3,5}^{\infty} b_n \cos\frac{n\pi x}{2a} Y_n\right)\right)$$

$$= \frac{\partial}{\partial x}\left(b_n \sum_{n=1,3,5}^{\infty} -\sin\frac{n\pi x}{2a}\times\frac{n\pi}{2a}\times Y_n\right)$$

$$= -b_n \sum_{n=1,3,5}^{\infty} \cos\frac{n\pi x}{2a}\times\frac{n\pi}{2a}\times\frac{n\pi}{2a} Y_n$$

$$\frac{\partial^2 z}{\partial y^2} = \sum_{n=1,3,5}^{\infty} b_n \cos\frac{n\pi x}{2a}\times Y_n''$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{-q}{s}$$

$$\sum_{n=1,3,5}^{\infty} b_n \cos\frac{n\pi x}{2a} Y_n'' - \sum_{n=1,3,5} b_n \cos\frac{n\pi x}{2a}\times\frac{n^2\pi^2}{4a^2} Y_n = \frac{-q}{s}$$

$$b_n \sum_{n=1,3,5}^{\infty} \cos\frac{n\pi x}{2a}\left(Y_n'' - \frac{n^2\pi^2}{4a^2}Y_n\right) = -\frac{q}{s}$$

actually $\boxed{b_n \sum_{n=1,3,5}^{\infty} \cos\frac{n\pi x}{2a} = \frac{n\pi}{4} b_n (-1)^{(n-1)/2}} \rightarrow (c)$ → from series

so $\boxed{\frac{n\pi}{4} b_n (-1)^{(n-1)/2}\left(Y_n'' - \frac{n^2\pi^2}{4a^2}Y_n\right) = \frac{-q}{s}} \rightarrow (d)$

$$\boxed{Y_n'' - \frac{n^2\pi^2}{4a^2} Y_n = \frac{-q}{S} \times \frac{4}{n\pi b_n} (-1)^{(n-1)/2}} \rightarrow (e)$$

from which

$$\boxed{Y_n = A \sinh \frac{n\pi y}{2a} + B \cosh \frac{n\pi y}{2a} + \frac{16 q a^2}{S n^3 \pi^3 b_n} (-1)^{(n-1)/2}} \rightarrow (F)$$

⇒ From the condition of symmetry the deflection surface of the membrane with respect to x axis, it follows constant integration "A" must be zero.

⇒ The constant B is det. from the condition that the deflections of the membrane are zero for $y = \pm b$

i.e., $(Y_n)_{y=\pm b} = 0$

⇒ which gives

from (F)

$$\boxed{Y_n = 0 = B \cosh \cdot \frac{n\pi b}{2a} + \frac{16 q a^2}{S n^3 \pi^3 b_n} (-1)^{(n-1)/2}} \rightarrow (g)$$

$$B \cosh \frac{n\pi b}{2a} = -\frac{16 q a^2}{S n^3 \pi^3 b_n} (-1)^{(n-1)/2}$$

$$\boxed{B = \frac{-16 q a^2}{S n^3 \pi^3 b_n} (-1)^{(n-1)/2} \times \frac{1}{\cosh \frac{n\pi b}{2a}}} \rightarrow \text{substitute in } (g)$$

$$Y_n = \frac{-16 q a^2}{S n^3 \pi^3 b_n} (-1)^{(n-1)/2} \times \frac{1}{\cosh \frac{n\pi b}{2a}} \times \cosh \frac{n\pi y}{2a} + \frac{16 q a^2}{S n^3 \pi^3 b_n} (-1)^{(n-1)/2}$$

$$\boxed{Y_n = \frac{16 q a^2}{S n^3 \pi^3 b_n} (-1)^{(n-1)/2} \left[1 - \frac{\cosh \left(\frac{n\pi y}{2a}\right)}{\cosh \left(\frac{n\pi b}{2a}\right)} \right]} \rightarrow (h)$$

⇒ the general Expression for the deflection surface of the membrane

substitute (h) value in $z = \sum_{n=1,3,5}^{\infty} b_n \cos\frac{n\pi x}{2a} Y_n$

$$z = \sum_{n=1,3,5}^{\infty} \cancel{b_n} \times \cos\frac{n\pi x}{2a} \times \frac{16 q a^2}{S n^3 \pi^3 \cancel{b_n}} (-1)^{(n-1)/2} \left[1 - \frac{\cosh(n\pi y/2a)}{\cosh(n\pi b/2a)}\right]$$

$$\Rightarrow \boxed{z = \frac{16 q a^2}{S \pi^3} \sum_{n=1,3,5}^{\infty} \frac{1}{n^3} (-1)^{(n-1)/2} \left[1 - \frac{\cosh(n\pi y/2a)}{\cosh(n\pi b/2a)}\right] \cos\frac{n\pi x}{2a}} \rightarrow (i)$$

⇒ Replacing q/S by $2G\theta$, we obtain for the stress function.

$$\phi = \frac{32 G \theta a^2}{\pi^3} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^3} (-1)^{(n-1)/2} \left[1 - \frac{\cosh(n\pi y/2a)}{\cosh(n\pi b/2a)}\right] \cos\frac{n\pi x}{2a}$$

⇒ here ϕ means stress function if u seen. that

$$\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial x^2} = F = -2G\theta \qquad \phi = \text{stress function}$$

Same. here z = stress function we can take z as ϕ

⇒ The stress components are obtained from Eqs

$$\tau_{yz} = -\frac{\partial\phi}{\partial x} = \cancel{\frac{16 G\theta a}{\pi^2}}$$

$$= \frac{\overset{16}{\cancel{32}} G\theta \times \cancel{2} a^{\cancel{2}}}{\pi^{\cancel{3}2}} \sum_{n=1,3,5}^{\infty} \frac{1}{n^{\cancel{3}2}} \times (-1)^{(n-1)/2} \left[1 - \frac{\cosh(n\pi y/2a)}{\cosh(n\pi b/2a)}\right] \sin\frac{n\pi x}{2a} \times \frac{\cancel{n\pi}}{\cancel{2a}}$$

$$\boxed{\tau_{yz} = \frac{16 G\theta a}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} (-1)^{(n-1)/2} \left[1 - \frac{\cosh(n\pi y/2a)}{\cosh(n\pi b/2a)}\right] \sin\frac{n\pi x}{2a}} \rightarrow (j)$$

⇒ Assuming that $b > a$, the max. shearing stress, corresponding to the max. slope of the membrane, is at the middle points of the long sides $x = \pm a$ of the rectangle substituting $x=a$; $y=0$ in (5)

$$\tau_{max} = \frac{16G\theta a}{\pi^2}\sum_{n=1,3,5,\dots}^{\infty}\frac{1}{n^2}(-1)^{(n-1/2)}\left[1-\frac{1}{\cosh(n\pi b/2a)}\right]$$

⇒ Let us τ_{max} as a function of torque in the form of (M_t)

$$M_t = 2\iint \phi\, dx\cdot dy \quad \text{from 6(a) in this unit}$$

$$= 2\int_{-a}^{a}\int_{-b}^{b}\phi\, dx\cdot dy$$

$$= \frac{64G\theta a^2}{\pi^3}\int_{-a}^{a}\int_{-b}^{b}\left\{\sum_{n=1,3,5\dots}^{\infty}\frac{1}{n^3}(-1)^{(n-1)/2}\left[1-\frac{\cosh(n\pi y/2a)}{\cosh(n\pi b/2a)}\right]\cos\frac{n\pi x}{2a}\right\}dx\cdot dy$$

* Solution of Torsional Problems by Energy Method:-

① We have seen that the solution of torsional problems is reduced in each particular case of the stress function satisfying the diff. eqn $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=F$ &

Boundary condition $\frac{\partial\phi}{\partial y}\frac{dy}{ds}+\frac{\partial\phi}{\partial x}\frac{dx}{ds}=\frac{d\phi}{ds}=0$

② In deriving an approximate solution of the problem it is useful, instead of working with the diff. equation, to det. stress function from the min. condition of a certain integral, which can be obtained from consideration of the strain energy of the twisted bar

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③ The strain energy of the twisted bar per unit length,

$V = \int \frac{\tau^2}{2G} dx$ (For a twisted bar) actually in two sides shear stress so that

τ^2 becomes $\tau_{xz}^2 + \tau_{yz}^2$

$$V = \iint \left(\frac{\tau_{xz}^2 + \tau_{yz}^2}{2G}\right) . dx \cdot dy$$

$$\boxed{V = \frac{1}{2G} \iint \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right] dx \cdot dy.} \rightarrow (K)$$

④ if we give to the stress function ϕ any small variation $\delta\phi$ the ~~variation in~~ torque is also varied

i.e.,

$$V = \frac{1}{2G} \iint \delta \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right] dx \cdot dy$$

$$\boxed{V = \frac{\delta}{2G} \iint \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right] dx \cdot dy.} \rightarrow (L)$$

~~the torque~~ M_t as also varied so

~~$M_t = 2\iint \delta\phi \cdot dx \cdot d$~~

actually $M_t = 2\iint \phi \, dx \cdot dy$ it becomes $\boxed{M_t = 2\iint \delta\phi \, dx \cdot dy} \rightarrow (M)$

~~Eqn~~ Equating (L) & (M) $\times \theta$ i.e., $\boxed{\text{S.E} = \text{Torque developed by body}}$

$$\frac{\delta}{2G} \iint \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right] dx \cdot dy = 2\theta \iint \delta\phi \, dx \cdot dy.$$

(or)

$$\delta \iint \left[\frac{1}{2}\left(\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right)\right] dx \cdot dy = 2G\theta \iint \delta\phi \, dx \cdot dy.$$

(or)

$$\delta \iint \left\{\frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right]\right\} dx \cdot dy - 2G\theta \iint \delta\phi \, dx \cdot dy = 0$$

the S.E will "U"

$$\boxed{U = \iint \left\{ \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right] - 2G\theta\phi \right\} dx \cdot dy} \rightarrow (11)$$

(5) We come also to the same conclusion by using Membrane analogy & Principle of Virtual work.

if "S" is the uniform tension in the membrane, the increase in strain energy of membrane due to deflection is obtained by Multiplying the tension S by the increase of surface of the membrane

$$\boxed{\frac{1}{2} S \iint \left[\left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right] dx \cdot dy} \rightarrow (12)$$

(6) Where z is the ~~Membrane~~ deflection of the Membrane

if We take now a Virtual displacement of the membrane from the position of equilibrium, the change in the S.E of membrane due to this displacement must be equal to the work done by the uniform load "q" on the virtual displacement.

$$\frac{1}{2} S \delta \iint \left[\left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right] dx \cdot dy = \iint q \cdot \delta z \cdot dx \cdot dy$$

gives

$$\iint \left\{ \left[\frac{1}{2} \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right] - \frac{q}{S} z \right\} dx \cdot dy$$

if We substitute in this Integral $2G\theta$ for q/S we arrive same (11)th eqn